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Thermodynamics of theories with sixteen supercharges in non-trivial vacua

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Abstract: We study the thermodynamics of maximally supersymmetric $\mathrm{U}(N)$ Yang-Mills theory on $\mathbb{R} \times S^2$ at large N. The model arises as a consistent truncation of $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R} \times S^3$ and as the continuum limit of the plane-wave matrix model expanded around the N spherical membrane vacuum. The theory has an infinite number of classical BPS vacua, labeled by a set of monopole numbers, described by dual supergravity solutions. We first derive the Lagrangian and its supersymmetry transformations as a deformation of the usual dimensional reduction of $\mathcal{N} = 1$ gauge theory in ten dimensions. Then we compute the partition function in the zero 't Hooft coupling limit in different monopole backgrounds and with chemical potentials for the R-charges. In the trivial vacuum we observe a first-order Hagedorn transition separating a phase in which the Polyakov loop has vanishing expectation value from a regime in which this order parameter is non-zero, in analogy with the four-dimensional case. The picture changes in the monopole vacua due to the structure of the fermionic effective action. Depending on the regularization procedure used in the path integral, we obtain two completely different behaviors, triggered by the absence or the appearance of a Chern-Simons term. In the first case we still observe a first-order phase transition, with Hagedorn temperature depending on the monopole charges. In the latter the large N behavior is obtained by solving a unitary multi-matrix model with a peculiar logarithmic potential, the system does not present a phase transition and it always appears in a "deconfined" phase.

Keywords: 1/N Expansion, Confinement, AdS-CFT Correspondence, Gauge-gravity correspondence.

Contents

1. Introduction

In the context of the AdS/CFT correspondence [1−4] an interconnected family of theories with sixteen supercharges has been recently studied [5]. They all have a mass gap and a discrete spectrum of excitations. These theories can be obtained from consistent truncations of $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R} \times S^3$ and have many BPS vacua. Remarkably, smooth gravity solutions corresponding to all these vacua can be described rather explicitly. At large 't Hooft coupling some properties of the dual string theory have also been examined according to the pioneering proposal of [6].

From the gauge theoretical point of view it seems particulary appealing to investigate the properties of one specific theory belonging to this class, namely the maximally supersymmetric $\mathrm{U}(N)$ Yang-Mills theory on $\mathbb{R}\times S^2$. This theory already appeared in [7] where it arises from the fuzzy sphere vacuum (membrane vacuum) of the plane-wave matrix model by taking a large N limit that removes the fuzzyness. The model can also be constructed from the familiar $\mathcal{N} = 4$ SYM theory by truncating the free-field spectrum on $\mathbb{R} \times S^3$ to states that are invariant under $\mathrm{U}(1)_L \subset \mathrm{SU}(2)_L$, where $\mathrm{SU}(2)_L$ is one of the SU(2) factors in the SO(4) rotation group of the three-sphere. Geometrically this corresponds to a dimensional reduction of the four-dimensional supersymmetric theory along the U(1) fiber of S^3 seen as an Hopf fibration over S^2. The resulting model lives in one dimension less and maintains supersymmetry through a rather interesting mechanism. The particular dimensional reduction breaks the natural SO(7) R-charge symmetry to SO(6), singling out one of the seven scalars of the maximally supersymmetric Yang-Mills theory, which then behaves differently from the others. It combines with the gauge fields to form a peculiar Chern-Simons-like term that is crucial to preserve the sixteen supercharges, balancing the appearance of mass terms for fermions and scalars. The BPS vacua are generated by the same term that allows to combine the field strength and the scalar into a perfect square whose zero-energy configurations are determined by N integers $n_1, \ldots, n_N$ associated to monopole numbers on the sphere.

The model represents an interesting example of a supersymmetric non-conformal gauge theory, with smooth gravitational dual and non-trivial vacuum structure, defined on a compact space. The last feature is particulary appealing in the study of the thermal properties of the theory. Recently the thermodynamics of large N theories on compact spaces has attracted much attention. On compact spaces the Gauss's law restricts physical states to gauge singlets. Consequently, even at weak 't Hooft coupling the theories are in a confining phase at low temperature and undergo a deconfinement transition at a critical temperature. For example, the partition function of $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R} \times S^3$ was computed at large N and small coupling in [8−10]. It was shown that the free energy is of order $\mathcal{O}(1)$ at low temperature and of order $\mathcal{O}(N^2)$ above a critical temperature. At strictly zero 't Hooft coupling the transition is a first-order Hagedorn-like transition. At small coupling a first or a second order transition is expected, depending on the particular matter content of the theory. The computation in the $\mathcal{N}=4$ maximally supersymmetric case has never been performed but in [11] it was argued that the maximally supersymmetric plane-wave deformation of Matrix theory and $\mathcal{N}=4$ SYM should show similar behavior, including thermodynamics. The plane wave matrix model is a theory with sixteen supercharges and it was argued in [7] to be dual to a little string theory compactified on S^5. For a small sphere, this theory is weakly coupled and one may study the little string theory thermodynamics rather explicitly [12]. The phase transition for this model was shown to remain first order in [13] indicating that this might also be the case for $\mathcal{N}=4$ SYM. This was shown by computing the relevant parts of the effective potential for the Polyakov loop operator to three loop order [13]. With the same procedure it was shown in [14] that also for pure Yang-Mills the phase transition remains first-order up to three loops. The phase transition at weak coupling is basically driven by a Hagedorn-like behavior of the spectrum in the confining phase, suggesting a possible relationship with the dual description of large N gauge theories in terms of strings. For $\mathcal{N}=4$ the relevant string theory lives on an asymptotic AdS space and, at large 't Hooft coupling, the deconfinement phase transition corresponds to a Hawking-Page transition [15, 16]. The thermal AdS space dominates at low temperature and the AdS-Schwarzschild black hole is the relevant saddle-point in the high-temperature regime. The original proposal presented in [8, 9] to connect the phase transitions at small coupling on compact spaces with the gravitational/stringy physics stimulated a large number of investigations. Lower-dimensional theories on tori were examined in [17, 18], while the inclusion of chemical potentials for the R-charges was discussed in [19, 20] and, more recently, pure Yang-Mills theory on S^2 [21] was found to have a second order phase transition at small 't Hooft coupling.

In this paper we study the thermodynamics of $\mathcal{N}=8$ super Yang-Mills theory on $\mathbb{R}\times S^2$. We first derive the Lagrangian and its supersymmetry transformations as a deformation of the usual dimensional reduction of $\mathcal{N}=1$ gauge theory in ten dimensions. Actually our procedure will generate a larger class of three-dimensional theory: according to the particular choice of the generalized Killing spinor equation we obtain also theories on AdS_3 with peculiar Chern-Simons couplings. Then we compute the $\mathcal{N}=8$ partition function in the zero 't Hooft coupling limit, for different monopole vacua. In the trivial vacuum we observe a first-order Hagedorn transition separating a phase in which the Polyakov loop has vanishing expectation value from a regime in which this order parameter is non-zero, in complete analogy with the four-dimensional case. The Hagedorn temperature is also obtained in the presence of chemical potentials for the R-charges. Discussions on the dual gravitational picture [5] and the possibility of matching the gauge theory Hagedorn transition with a stringy Hagedorn transition, by exploiting for example a decoupling limit as in [20, 22 − 24] postponed to a forthcoming investigation.

The situation is very different in the non-trivial monopole vacua. The original $\mathrm{U}(N)$ gauge group is broken to a direct product $\mathrm{U}(N_1)\times\mathrm{U}(N_2)\times\ldots\mathrm{U}(N_k)$ and the constituent fields transform, in general, under bifundamental representations of $\mathrm{U}(N_I)\times\mathrm{U}(N_J)$. Because of the Gauss's law on a compact manifold, however, the only allowed excitations are $\mathrm{SU}(N_I) \times \mathrm{SU}(N_J)$ singlets. Different selection rules are instead possible for the $\mathrm{U}(1)$ charges in three dimensions, depending on the definition of the fermionic Fock vacuum in the presence of background monopoles [25]. The appearance of fermionic zero-modes makes possible, in general, to assign a non-trivial charge to the Fock vacuum, as clearly explained in [26]. In the path-integral formalism this corresponds to precise choices in regularizing fermionic functional determinants which might produce Chern-Simons terms in the effective action. In our case the different possibilities are clearly manifested in the matrix model describing the partition function. We recall that, in the trivial vacuum, the thermal partition function is reduced to an integral over a single $\mathrm{U}(N)$ matrix [8, 9]

$$\mathcal{Z}(\beta) = \int [dU] \exp\Big[-S_{\text{eff}}(U)\Big] \tag{1.1}$$

where $U = e^{i\beta\alpha}$ (α is the zero mode of the gauge field A_0 on $S^2 \times S^1$ and $\beta = 1/T$ the inverse of the temperature). In the non-trivial monopole vacuum $\mathcal{Z}(\beta)$ is given instead by a multi-matrix model over a set of unitary matrices $U_I(N_I)$, $i = 1, 2, \ldots k$, reflecting the breaking of the $\mathrm{U}(N)$ gauge group. More importantly the effective action $S_{\text{eff}}(U_I)$, at zero 't Hooft coupling, can be modified by the presence of logarithmic terms $NQ_I \mathrm{Tr} \log(U_I)$ that implement selection rules on the $\mathrm{U}(1)$ charges. The large N analysis is highly affected by these new interactions: they contribute at order N^2 and can drive the relevant saddle-point always at a non-zero value of the Polyakov loop. Unitary matrix model of the kind we encountered in our analysis have been previously considered in the eighties [27, 28], but with an important difference: in those studies the coefficient weighting the logarithmic term Tr $\log(U)$ in the action was taken independent on N. Conversely the large N saddle-points were not modified by its presence, being determined by the rest of the action. In our case, instead, we have to cope with a linear dependence on N and we cannot simply borrow those results. We have therefore performed an entirely new large N analysis of these kind of models, starting from an exact differential equation of the Painlevé type that describes the finite N partition function [29].

The paper is organized as follows. In section 2 we construct the supersymmetric Yang-Mills theory on $\mathbb{R} \times S^2$ using a different strategy with respect to [5] and [7] (see also [30] for a careful derivation of the Hopf reduction and [31] for an extension to more general fiber bundles). We start from $\mathcal{N} = 1$ super Yang-Mills theory in ten dimensions and consider its dimensional reduction on $\mathbb{R} \times S^2$. We find the relevant Killing spinors that generate the rigid supersymmetry, generalizing to our case the approach developed in [32]. We further determine the deformations of the original ten dimensional Lagrangian and of the supersymmetry transformations ensuring the global invariance of the action. Interestingly, using the same strategy it is possible to construct two other maximally supersymmetric gauge theories on three-dimensional curved spacetimes, living both on AdS_3 and differing from the theory introduced in [7] in the structure of the Chern-Simons terms. In section 3 we briefly examine the BPS vacua of the model, we comment on their gravitational description and the related instanton solutions.

We then turn to study the thermodynamics at zero 't Hooft coupling. Following the analysis in [8, 9], we obtain the partition function of the theory in a generic vacuum, in terms of matrix integrals. In section 4 we present the results of the relevant functional determinants in the background of a gauge flat-connection and of a monopole potential, recovering the appropriate single-particle partition functions for scalars, spinors and vectors. Careful ζ-function evaluations are deferred to the appendices. We discuss the emerging, on the monopole background, of new logarithmic terms in the effective action, directly related, in this formalism, to the appearance of fermionic zero-modes. We explain their dependence on the regularization procedure and remark their interplay with a typical three-dimensional phenomenon, the induction of Chern-Simons terms. We interpret their effect as a part of the projection into singlets of the gauge group, as required by the Gauss's law. Section 5 is devoted to discuss the large N thermodynamics in the trivial vacuum. We determine the critical temperature at which the first-order phase transition takes place and we generalize the result to the case of non vanishing chemical potentials for the R-charges. Finally, in sections 6 and 7, we study the large N theory on the non-trivial monopole backgrounds: we consider a large class of vacua, characterized by the set of integers $n_1, \ldots, n_k$ and large N degeneracies $N_1, \ldots, N_k$. According to the discussion of section 4, we study two different choices for the logarithmic terms, within our regularization procedure. First, in section 6, we discuss the "uncharged" case, that amounts to make a particular choice of branch cuts, in the ζ-function regularization procedure [33, 34], that cancels the Chern-Simons like contributions. In turn we get a non-vanishing Casimir energy, depending explicitly on the monopole background. The resulting unitary multi-matrix model is an obvious generalization of the trivial case. We find again a first-order phase transition, with an Hagedorn temperature explicitly depending on the monopole numbers. We discuss also some particular class of vacua, characterized by large monopole charges, whose Hagedorn temperature approaches the one of the theory on $S^3/\mathbb{Z}_k$ in trivial vacuum. In section 7 we discuss the opposite situation of a "maximally" charged fermionic vacuum: we have a non-trivial modification of the unitary multi-matrix model due to appearance of the new logarithmic terms and vanishing Casimir energy. For the sake of clarity we will restrict our discussion to a particular simple background $(n, n, \ldots, n, -n, -n \ldots, -n)$. We show the existence of a non-trivial saddle-point for the effective action for a wide range of temperatures starting from zero, within the assumption that we can disregard higher windings contributions in this regime. This implies that the theory is always in a "deconfined" phase. We have to face the problem of computing the free energy and the phase structure of the matrix model

$$\mathcal{Z}(\beta, p) = \int DU \exp\left(\beta N(\mathrm{Tr}(U) + \mathrm{Tr}(U^\dagger))\right) \det(U)^{Np}, \tag{1.2}$$

that is a non-trivial deformation of the familiar Gross-Witten model [35]. Its large N behavior is carefully studied in section 7.1 , obtaining the exact free energy in terms of the solution of a fourth-order algebraic equation: we prove that there is no phase transition as long as $p \neq 0$, in contrast with the usual $p = 0$ case, that appears as a singular point in the parameter space. In section 7.2 we use the results of our analysis to derive a set of saddle-point equations for the partition function which describes the "deconfined" phase. The disappearance of the confining regime is consistent with the known results on finite temperature 2+1 dimensional gauge theories where, once a topological mass (a

Chern-Simons term) is turned on, there cannot be a phase transition [36 − 38]. In section 8 we briefly draw our conclusions and discuss future directions. Several appendices are devoted to technical aspects and to an alternative derivation of the partition functions. In appendix A we report some details on supersymmetry transformations. In appendix B we give the details of the computation of functional determinants. In appendix C we recover the results for the single-particle partition functions from those of the parent $\mathcal{N} = 4$ theory by explicitly constructing the projector into the U(1) invariant modes. We also check the consistency of our results with those of [39], where the theory on $\mathbb{R} \times S^3/\mathbb{Z}_k$ has been studied. Appendix D is instead focused on some technical aspects, related to the solution of the large N matrix integrals.

2. Lagrangian and supersymmetry on $\mathbb{R} \times S^2$ from $D = 10$

There are many ways to construct the Lagrangian of the gauge theory with sixteen supercharges on $\mathbb{R} \times S^2$ and its supersymmetry transformations. For instance, in [7] this theory was obtained from the plane-wave matrix model action expanded around the k-membrane vacuum in the large N limit. Subsequently, in [5] it was derived as a U(1) truncation of the spectrum of the $\mathcal{N} = 4$ gauge theory on $\mathbb{R} \times S^3$. Since here we shall be mainly concerned with the field theoretical features of this $\mathcal{N} = 8$ model, we shall follow a more conventional (and maybe pedagogical) approach: the Lagrangian and its supersymmetry transformations will be derived as a *deformation* of the standard toroidal compactification of $\mathcal{N} = 1$ gauge theory in ten dimensions.

We first consider the theory on the flat Minkowski space in three dimensions, $\mathbb{M}_{(1,2)}$. The $\mathcal{N} = 8$ theory in this case is the straightforward dimensional reduction of the $\mathcal{N} = 1$ theory in $D = 10$. The most convenient and compact way to present its Lagrangian is to maintain the ten-dimensional notation and to write (see appendix A for a summary of our conventions[1])

$$\mathcal{L}^{(0)} = -\frac{1}{2} F_{MN} F^{MN} + i\overline{\psi}\Gamma^M D_M \psi. \tag{2.1}$$

All the fields in (2.1) only depend on the space-time coordinates (x^0, x^1, x^2). In particular, from the three-dimensional point of view, the gauge field A_M contains the reduced gauge field A_μ and seven scalars $(\phi_m) = (\phi_3, \phi_4, \cdots, \phi_9) \equiv (\phi_3, \phi_{\overline{m}})$. The flat ten dimensional space-time metric is diagonal and it has the factorized structure $\mathbb{T}^7 \times \mathbb{M}_{(1,2)}$.

Our goal is now to promote the supersymmetric theory in the flat 2+1-dimensional space-time to a supersymmetric theory on the curved space $\mathbb{R} \times S^2$. It is useful to keep a ten-dimensional notation where the above space-time is viewed as a submanifold embedded in $\mathbb{T}^7 \times \mathbb{R} \times S^2$ with the metric

$$ds^2 = -dt^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) + \sum_{i=1}^{7} d\eta_i^2 \ . \tag{2.2}$$

[1] In general we shall omit the trace over the gauge generator in our equations, unless it is source of confusion.

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Here the coordinates θ and φ span the sphere S^2 of radius R, while the internal angular coordinates η_i parameterize the torus $\mathbb{T}^7$. The action (2.1) in the background (2.2) is still meaningful once we introduce the appropriate dependence on the vielbein and the spin-connections in the covariant derivatives. The real issue is whether this theory will have any supersymmetry. The action (2.1) on flat space is invariant under the usual supersymmetry transformations written in terms of a constant arbitrary spinor ϵ

$$\begin{aligned} \delta^{(0)} A_M &= -2i\overline{\psi}\Gamma_M\epsilon, \\ \delta^{(0)}\psi &= F_{MN}\Gamma^{MN}\epsilon\ . \end{aligned} \tag{2.3}$$

Constant spinors however do not exist, in general, on a curved space. For a space-time of the type (2.2), the notion of a constant spinor should be replaced with that of a *Killing spinor,Blau:2000xg*. Its specific definition may depend on the detail of the geometry, but, for us, it will be a spinor satisfying an equation of the type

$$\nabla_\mu\epsilon = K_\mu{}^\nu\Gamma_\nu\Gamma^{123}\epsilon\ , \tag{2.4}$$

where the Greek indices run only over the three-dimensional space-time since the transverse coordinates η_i are flat and we can always choose ϵ to be a constant along these directions. In (2.4) we have also inserted an additional dependence on the Γ matrices through a monomial factor Γ^{123}.[2] This has double role: (a) it makes (2.4) compatible with the ten-dimensional chirality conditions; (b) it generates, as we shall see, the relevant massive deformations for our fields. Finally the tensor $K_\mu{}^\nu$ expresses an additional freedom in constructing the Killing spinors. In a curved space, there is in fact no *a priori* reason to treat all the coordinates symmetrically. In the $\mathbb{R}\times S^2$ curved space-time geometry there is a natural splitting between space and time and thus it is quite natural to weight them differently by choosing

$$K_\mu{}^\nu = \alpha\left[\left(\delta_\mu^\nu + k_\mu k^\nu\right) - \mathcal{B}k_\mu k^\nu\right], \tag{2.5}$$

where k_μ is the time-like Killing vector of (2.2) and $\alpha, \mathcal{B}$ are two arbitrary parameters. The parameter α is fixed by imposing the necessary integrability condition (the first) [40], which arises from the commutator $[\nabla_\mu, \nabla_\nu]\epsilon$. This can be either expressed in terms of the space-time curvature scalar $\mathcal{R} = 2/R^2$ or, through (2.4), in terms of $K_\mu{}^\nu$ and consequently of α. We thus get for α

$$\alpha = \frac{1}{2R}\ . \tag{2.6}$$

The parameter $\mathcal{B}$, instead, remains free and it will be determined in the following.

The variation of the action (2.1) with respect to the supersymmetry transformations (2.3) written in terms of a non-constant supersymmetry parameter ϵ does not vanish. Terms depending on the covariant derivatives of ϵ (2.4) are in fact generated (see ap-

[2] The direction $(1, 2)$ span the tangent space to the sphere S^2, while the index 3 is along the first of the compactified dimensions.

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pendix A for conventions and more details)

$$\begin{aligned}\delta^{(0)}\mathcal{L}^{(0)} &= 2\mathbb{R}\mathrm{e}\{i\overline{\psi}F_{MN}\Gamma^{\mu}\Gamma^{MN}\nabla_{\mu}\epsilon\}\\ &= 2\mathbb{R}\mathrm{e}\{i\mathcal{B}\alpha\overline{\psi}[\Gamma^{ij}F_{ij}-2\Gamma^{0i}F_{0i}+2\Gamma^{jm}D_j\phi_m-2\Gamma^{0m}D_0\phi_m-ig\Gamma^{mn}[\phi_m,\phi_n]]\Gamma^{123}\epsilon\\ &\quad+i\alpha\overline{\psi}[-2\Gamma^{ij}F_{ij}+4\Gamma^0D_0\phi_m-2ig\Gamma^{mn}[\phi_m,\phi_n]]\Gamma^{123}\epsilon\}.\end{aligned} \tag{2.7}$$

where in the second equality we have used (2.4) and (2.5). This undesired variation can be compensated by adding the following deformations to the original Lagrangian

$$\mathcal{L}^{(1)}=iM\alpha\overline{\psi}\Gamma^{123}\psi+N\alpha\phi_3F_{12},\qquad \mathcal{L}^{(2)}=V\alpha^2\phi_m^2+W\alpha^2\phi_3^2, \tag{2.8}$$

and by adding new terms to the supersymmetry transformations of the fermions

$$\delta^{(1)}\psi=P\alpha\Gamma^m\Gamma^{123}\phi_m\epsilon+G\alpha\Gamma^3\Gamma^{123}\phi_3\epsilon, \tag{2.9}$$

where M,N,V,W,P,G are arbitrary parameters to be fixed by imposing the invariance of the complete action. The size of the deformations is tuned by the natural mass scale $\alpha=1/(2R)$ provided by the radius of the sphere.

Some comments on the form of (2.8) and (2.9) are in order. The addition of mass terms for the scalars ($\mathcal{L}^{(2)}$) is a common and well-known property for supersymmetric theories in a background admitting Killing spinors. Some of the mass terms can also be justified with the requirement that the conformal invariance originally present in flat space is preserved. In four dimensions, for $\mathcal{N}=4$ super Yang-Mills, this is the only required modification of the Lagrangian because of an accidental cancellation. Since we are in three dimensions, we are also forced to introduce a non-standard mass term for the fermions (the first term in $\mathcal{L}^{(1)}$). The natural supersymmetric companion for a fermionic mass in $D=3$ is then a Chern-Simons-like term (the second term in $\mathcal{L}^{(1)}$). Its unusual form, ϕ_3F_{12}, mixes the scalar ϕ_3 with the gauge-fields and is inherited from the particular choice of the monomial Γ^{123} in (2.4). Then the modifications (2.9) in the supersymmetry transformations are the only possible ones with the right dimensions and compatible with the symmetries of the theory.

The most convenient and simple way to analyze the effect of the additional terms in the Lagrangian (2.8) and in the supersymmetry transformations (2.9) is to single out, in the variation of the Lagrangian, different powers of the deformation parameter α. We start with the linear order in α, the zeroth order being automatically absent since our theory is supersymmetric in flat space-time. At this order we have three contributions: the original variation (2.7), the variation of the new Lagrangian $\mathcal{L}^{(1)}$ with respect to the old transformations (2.3)

$$\begin{aligned}\delta^{(0)}\mathcal{L}^{(1)}=&2M\alpha\mathbb{R}\mathrm{e}\{i\overline{\psi}(F_{ij}\Gamma^{ij}-2F_{0i}\Gamma^{0i}-2D_0\phi_3\Gamma^{03}+2D_i\phi_3\Gamma^{i3}+2D_0\phi_{\overline{m}}\Gamma^{0\overline{m}}-2D_i\phi_{\overline{m}}\Gamma^{i\overline{m}}+\\ &+2i[\phi_3,\phi_{\overline{m}}]\Gamma^{3\overline{m}}-i[\phi_{\overline{m}},\phi_{\overline{n}}]\Gamma^{\overline{m}\,\overline{n}})\Gamma^{123}\epsilon\}+iN\alpha(F_{ij}\overline{\psi}\Gamma^{ij}+2D_i\phi_3\overline{\psi}\Gamma^{i3})\Gamma^{123}\epsilon\end{aligned} \tag{2.10}$$

and finally the variation of $\overset{(0)}{\mathcal{L}}$ with respect to (2.9)

$$\overset{(1)}{\delta}\overset{(0)}{\mathcal{L}}=2\mathbb{R}\mathrm{e}\{i\alpha\overline{\psi}(P\Gamma^{\mu m}D_\mu\phi_m-igP\Gamma^{mn}[\phi_m,\phi_n]+G\Gamma^{\mu 3}D_\mu\phi_3-igG\Gamma^{m3}[\phi_m,\phi_3])\Gamma^{123}\epsilon\}. \tag{2.11}$$

See appendix A for all the different index conventions. It is quite straightforward to derive (2.10) and (2.11) since at this order in α we can consider ϵ as a constant spinor, namely $\nabla_\mu\epsilon=0$. Imposing that $\overset{(0)}{\delta}\overset{(0)}{\mathcal{L}}+\overset{(0)}{\delta}\overset{(1)}{\mathcal{L}}+\overset{(1)}{\delta}\overset{(0)}{\mathcal{L}}=\mathcal{O}(\alpha^2)$ gives a linear system of eight equations in the five unknowns M, N, P, G and $\mathcal{B}$. The details are given in appendix A.1. Quite surprisingly, this system is still solvable and it fixes the value of the above constants as

$$M=-\frac{1}{2},\qquad N=4,\;\; P=-2,\qquad G=-2,\qquad \mathcal{B}=\frac{1}{2}. \tag{2.12}$$

The next and final step is to consider the order α^2 in our supersymmetry variation. The situation is much simpler now since we need to evaluate only few terms. We have in fact to consider the effects of the corrected transformation (2.9) on $\overset{(1)}{\mathcal{L}}$

$$\overset{(1)}{\delta}\overset{(1)}{\mathcal{L}}=iM\alpha\overset{(1)}{\delta}(\overline{\psi}\Gamma^{123}\psi)=2\mathbb{R}\mathrm{e}\{i\alpha^2\overline{\psi}(\Gamma^{\overline{m}}\phi_{\overline{m}}-2\Gamma^3\phi_3)\epsilon\} \tag{2.13}$$

and we have to take care of the terms coming from $\overset{(1)}{\delta}\overset{(0)}{\mathcal{L}}$ originated from the covariant derivative of the Killing spinor ϵ. We obtain

$$\overset{(1)}{\delta}\overset{(0)}{\mathcal{L}}=-2\mathbb{R}\mathrm{e}\{i\alpha^2\overline{\psi}[3\Gamma^{\overline{m}}\phi_{\overline{m}}+6\Gamma^3\phi_3]\epsilon\}. \tag{2.14}$$

These two contributions are easily compensated by the variation of $\overset{(2)}{\mathcal{L}}$,

$$\delta^{(0)}\overset{(2)}{\mathcal{L}}=-4i\alpha^2(V\phi_{\overline{m}}\overline{\psi}\Gamma^{\overline{m}}\psi+(V+W)\phi_3\overline{\psi}\Gamma^3\psi)\,, \tag{2.15}$$

By setting $V=-1$ and $W=-3$ no surviving term is left! We remark that there is no $\mathcal{O}(\alpha^3)$ term, because there is neither an α-dependent term in the variation of bosons (which might produce a $\mathcal{O}(\alpha^3)$ term in the variation of $\overset{(2)}{\mathcal{L}}$) nor α^2 term in the variation of fermions.

We have thus reached our original goal: to promote the $\mathcal{N}=8$ theory in flat space in three dimensions to an $\mathcal{N}=8$ theory in the curved background $\mathbb{R}\times S^2$. Its Lagrangian in a ten-dimensional language is thus given by

$$\mathcal{L}=-\frac{1}{2}F_{MN}F^{MN}+i\overline{\psi}\Gamma^M D_M\psi-i\frac{\mu}{4}\overline{\psi}\Gamma^{123}\psi+2\mu\phi_3F_{12}-\frac{\mu^2}{4}\phi^2_{\overline{m}}-\mu^2\phi_3^2, \tag{2.16}$$

and it is invariant under the supersymmetry transformations

$$\begin{aligned}\delta A_M&=-2i\overline{\psi}\Gamma_M\epsilon,\\ \delta\psi&=F_{MN}\Gamma^{MN}\epsilon-\mu\Gamma^m\Gamma^{123}\phi_m\epsilon-\mu\Gamma^3\Gamma^{123}\phi_3\epsilon,\end{aligned} \tag{2.17}$$

where μ is the mass-scale $\mu=1/R$. Notice that the mass for the scalars $\phi_{\overline{m}}$ (with $\overline{m}=4,5,\ldots,9$) in (2.16) is that required by conformal invariance on $\mathbb{R}\times S^2$: $m^2_{conf.}=\frac{\mathcal{R}}{8}=$

$\frac{2}{8R^2} = \frac{\mu^2}{4}$. The mass of the scalar ϕ_3 is, instead, different because ϕ_3 mixes with the gauge fields. This mixing also breaks the original SO(7) R-symmetry present in flat space to the smaller group $\mathrm{SO}(6)_R$ ($\simeq \mathrm{SU}(4)_R$): the bosonic symmetries $\mathbb{R} \times \mathrm{SO}(3) \times \mathrm{SO}(6)_R$ combine with the supersymmetries into the supergroup SU(2|4). We have to mention that our presentation heavily relies on the general analysis of [32], where the problem of the existence of globally supersymmetric Yang-Mills theory on a curved space was addressed and some general recipes on how to construct these models were given. However, the Lagrangian (2.16) does not directly belong to the families of theories discussed in [32], it realizes nevertheless a straightforward generalization of them. We have in fact allowed for a more general Killing spinor equation both by including the additional matrix factor $K_\mu{}^\nu$ and by considering a monomial factor Γ^{123} mixing one of the transverse compact directions with the two spatial directions of the actual space-time of the theory.

The Lagrangian (2.16) written in terms of the three-dimensional fields becomes

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 2i\overline{\lambda}_i\gamma^\mu D_\mu\lambda^i - \frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij} - D_\mu\phi_3 D^\mu\phi_3 - 2ig\overline{\lambda}_i[\phi_3,\lambda^i] +\\ &- g\sqrt{2}\left(\lambda^{iT}[\phi_{ij},\varepsilon\lambda^j] - \overline{\lambda}_i[\phi^{ij},\varepsilon\overline{\lambda}_j^T]\right) + \frac{1}{8}g^2[\phi_{ij},\phi_{kl}][\phi^{ij},\phi^{kl}] + \frac{1}{2}g^2[\phi_3,\phi_{ij}][\phi_3,\phi^{ij}] +\\ &- \frac{\mu}{2}\overline{\lambda}_i\gamma^0\lambda^i - \frac{\mu^2}{8}\phi_{ij}\phi^{ij} - \mu^2\phi_3^2 + 2\mu\phi_3 F_{12}\ .\end{aligned} \tag{2.18}$$

This is the $\mathcal{N} = 8$ SYM Lagrangian on $\mathbb{R} \times S^2$ that will be used in computing the thermodynamic partition function of the model. We have cast the contribution of the scalar fields $(\phi_4, \ldots, \phi_9)$ in an $\mathrm{SU}(4)_R$ manifestly covariant form, by rewriting their Lagrangian in terms of the **6** representation of $\mathrm{SU}(4)_R$, ϕ_{ij}. The spinor fields λ_i are four Dirac spinors in $D = 3$ originating from the dimensional reduction of ψ.

Since we will be mainly interested in the finite temperature features of the model, the Euclidean version of (2.18) will be more relevant. It is given by

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\overline{\lambda}_i\gamma^\mu D_\mu\lambda^i + \frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij} + D_\mu\phi_3 D^\mu\phi^3 +\\ &+ g\sqrt{2}\left(\lambda^{iT}[\phi_{ij},\varepsilon\lambda^j] - \overline{\lambda}_i[\phi^{ij},\varepsilon\overline{\lambda}_j^T]\right) + 2ig\overline{\lambda}_i[\phi_3,\lambda^i] +\\ &- \frac{1}{8}g^2[\phi_{ij},\phi_{kl}][\phi^{ij},\phi^{kl}] - \frac{1}{2}g^2[\phi_3,\phi_{ij}][\phi_3,\phi^{ij}] +\\ &+ \frac{i\mu}{2}\overline{\lambda}_i\gamma^0\lambda^i + \frac{\mu^2}{8}\phi_{ij}\phi^{ij} + \mu^2\phi_3^2 - 2\mu\phi_3 F_{12}.\end{aligned} \tag{2.19}$$

We conclude by noting that, in the above analysis, we have made a particular choice in considering the form of the Killing spinor equation. A careful reader might wonder if there are other possibilities. Unfortunately, different choices in (2.4) generally lead to inconsistencies: the Killing equation is not integrable or no consistent supersymmetric deformation exists. For example, the second type of inconsistency would occur if we had simply chosen $K_\mu{}^\nu = \delta_\mu^\nu$. It is however intriguing to note that the choice $K_\mu{}^\nu = \delta_\mu^\nu$ becomes consistent if we alter the background geometry from $\mathbb{R} \times S^2$ to AdS_3 and substitute Γ^{123} with Γ^{012} or Γ^{456}. In the former case, we would have found a maximally supersymmetric version of the topologically massive theory, with bosonic symmetry group $\mathrm{SO}(1,3)\times\mathrm{SO}(7)$. In the latter we would have instead reached a massive deformation of the maximally supersymmetric Yang-Mills with the peculiar interaction $\mathrm{Tr}(\phi_3[\phi_4,\phi_5])$ and symmetry group $\mathrm{SO}(1,3)\times\mathrm{SO}(3)\times\mathrm{SO}(4)$. This case was already considered in [32]. It would be nice to understand better their relations with higher dimensional theories and to explore the possible existence of gravitational duals.

3. BPS vacua and their gravitational duals

In this section we shall briefly review the structure of the BPS vacua of the $\mathcal{N}=8$ theory on $\mathbb{R}\times S^2$ [5] that will be the main ingredients of the thermodynamical investigation of section 6 and 7. More specifically, we shall be interested in those vacua that maintain both the R-invariance and the geometrical symmetries.

In order to have an $\mathrm{SU}(4)_R$ invariant vacuum, we have to choose $\phi_{ij}=0$. Moreover, to preserve the invariance under time translations and the $\mathrm{SO}(3)$ rotations of the background geometry, we require that all the fields are time-independent and that the chromo-electric field $E_i=F_{0i}$ vanishes, respectively. The BPS condition can be derived from the requirement that on the supersymmetric invariant vacuum the supersymmetry variations should vanish. Fermions must be set to zero to saturate the BPS bound and consequently the supersymmetry variations of bosons automatically vanish on the vacuum. The supersymmetry variation of fermions, instead, must be set to zero and with the above assumptions it reads

$$0=\delta\psi=\left[2\left(F_{\theta\varphi}-\frac{1}{\mu}\sin\theta\phi_3\right)\Gamma^{\theta\varphi}+2D_\mu\phi_3\Gamma^{\mu3}\right]\epsilon\,, \tag{3.1}$$

(θ and φ are coordinates on S^2) which translates into two simple equations

$$F_{\theta\varphi}-\frac{1}{\mu}\sin\theta\phi_3=0,\qquad D_\mu\phi_3=0. \tag{3.2}$$

The reader familiar with YM_2 will immediately recognize in these equations, those of Yang-Mills theory on the sphere S^2, for which a complete classification of the solutions exists [41, 42]. The general solution for a $\mathrm{U}(N)$ theory is given by a stack of N independent $\mathrm{U}(1)$ Dirac monopoles of arbitrary charges. In detail, we have

$$\phi_3=\frac{\mu\mathfrak{f}}{2}\qquad F_{\theta\varphi}=\frac{\mathfrak{f}}{2}\sin\theta\qquad A=\frac{\mathfrak{f}}{2}\frac{(1-\cos\theta)}{\sin\theta}(\sin\theta d\varphi)\equiv\frac{\mathfrak{f}}{2}\mathcal{A}, \tag{3.3}$$

where $\mathfrak{f}$ is a diagonal matrix with integer entries, for which we shall use the short-hand notation

$$\mathfrak{f}=(n_1,N_1;n_2,N_2;\ldots;n_k,N_k). \tag{3.4}$$

Each n_I represents the Chern-class of the corresponding Dirac monopole and it assumes values in $\mathbb{Z}$, while N_I is the number of times that this charge appears on the diagonal. The vacuum (3.4) then breaks the original $\mathrm{U}(N)$ gauge symmetry to a direct product $\mathrm{U}(N_1)\times\mathrm{U}(N_2)\times\ldots\mathrm{U}(N_k)$. However, since all fields in (2.18) are in the adjoint representation, this breaking will affect the dynamics only through the relative charge (n_I-n_J) between different sectors, while the global charge $Q=\sum_{I=1}^k N_In_I$ will play no role.

The gravitational backgrounds dual to the vacua of these theories were derived in [5] and further discussed in [61] (where also the relations between vacua of theories with SU(2| 4) symmetry group are studied): they have an SO(3) and an SO(6) symmetry and thereby the geometry contains S^2 and S^5 factors, the remaining coordinates being time, a non-compact variable η, $-\infty \leq \eta \leq \infty$, and a radial coordinate ρ. These backgrounds are non-singular because the dual theories have a mass gap. The relevant supergravity equations can be reduced to a three-dimensional electrostatic problem where ρ is the radius of a charged disk. The ten dimensional metric and the other supergravity fields are completely specified in terms of the solution V of the related Laplace equation.[3] The regularity condition requires that the location where the S^2 shrinks are disks at constant η_i (in the ρ, η space) while S^5 shrinks along the segment of the $\rho = 0$ line between two nearby disks. The geometry therefore contains three-cycles connecting the shrinking S^2 and six-cycles connecting the shrinking S^5, supporting respectively non-trivial H_3 and $*F_4$ fluxes. There is a precise relation between these quantized fluxes and the data of the electrostatic problem, namely the electric charges Q_i of the disks are related to the RR fluxes while the distance (in the η direction) between two disks bounding a three cycle is proportional to the NS flux. To be more specific, this *electrostatic* description of a non-trivial vacuum generically contains k disks, whose positions are parameterized by k integers n_I through the relations

$$\eta_I = \frac{\pi n_I}{2}. \tag{3.5}$$

These integers are identified with the monopole charges n_I in (3.4). Moreover each disk carries a charge Q_I given by

$$Q_I = \frac{\pi^2 N_I}{8}, \tag{3.6}$$

where N_I are the same integer numbers counting the degeneracy of each monopole charge in the gauge theory. At the level of supergravity data, the above picture realizes k groups of $D2$ branes, each of N_I elements, wrapping different two-spheres. This is the geometric manifestation of the breaking of the gauge symmetry to a direct product $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_k)$. The charges n_I instead combine into NS5-fluxes given by $n_I - n_J$. Again the total charge seems to play no role.

In our field theoretical analysis we have neglected the time component of the gauge field A_0, which disappears from (3.2) when considering the solutions (3.3). Its dynamics is implicitly governed by the requirement that $E_i = 0$, which, for a time-independent background, becomes $D_i A_0 = 0$. It is a trivial exercise to show that the most general solution of this equation is provided by $A_0 = 0$ when the topology of the time direction is $\mathbb{R}$. In the finite temperature case where time is compactified to a circle S^1, the most general solution is, instead, given by $A_0 = a$, where a is a constant diagonal matrix, namely a flat-connection living on S^1. This will play a fundamental role in studying the thermodynamical properties of the theory.

It is instructive to look at the BPS vacua also at the level of the Euclidean Lagrangian: this will elucidate the emerging of an interesting class of instanton solutions thoroughly

[3] This problem has been recently tackled in [43] and [44], searching for a dual description of Little String theory on S^5

studied in [45]. If we focus on the bosonic sector of our model and we set $\phi_{ij} = 0$ to preserve the $\mathrm{SU}(4)_R$ symmetry, we can write

$$\sqrt{g}\mathcal{L} = \frac{\sqrt{g}}{2} F_{\alpha\beta}F^{\alpha\beta} + \sqrt{g}D_\alpha\phi_3 D^\alpha\phi_3 + \sqrt{g}\mu^2\phi_3^2 - 2\mu\phi_3 F_{\theta\varphi}. \tag{3.7}$$

This Lagrangian can be easily arranged in a *BPS-form*, i.e. as a sum of squares and total divergences. In fact, after some algebraic manipulation, the Euclidean Lagrangian can be cast in the following form

$$\begin{aligned}\sqrt{g}\mathcal{L} = &\pm\frac{1}{\mu}\sin\theta D_t(\phi_3^2) \mp D_\alpha(\phi_3 F_{\beta\rho}\epsilon^{\alpha\beta\rho}) + \sin\theta\left(F_{t\theta} \pm \frac{1}{\sin\theta}D_\varphi\phi_3\right)^2 + \\ &+\frac{1}{\sin\theta}\left(F_{t\varphi} \mp \sin\theta D_\theta\phi_3\right)^2 + \frac{\mu^2}{\sin\theta}\left(F_{\theta\varphi} - \frac{1}{\mu^2}\sin\theta(\mu\phi_3 \mp D_t\phi_3)\right)^2.\end{aligned} \tag{3.8}$$

Consequently, the minimum of the action is reached when the fields satisfy the following BPS-equations

$$(a):\ F_{t\theta} \pm \frac{D_\varphi\phi_3}{\sin\theta} = 0 \quad (b):\ F_{t\varphi} \mp \sin\theta D_\theta\phi_3 = 0 \quad (c):\ F_{\theta\varphi} - \frac{1}{\mu^2}\sin\theta(\mu\phi_3 \mp D_t\phi_3) = 0, \tag{3.9}$$

or in a compact and covariant notation

$$\sqrt{g}\epsilon_{\rho\nu\lambda}F^{\nu\lambda} = \mp 2D_\rho\phi_3 + 2\mu k_\rho\phi_3, \tag{3.10}$$

where k_ρ is the Euclidean version of the time-like Killing vector of the metric on $\mathbb{R} \times S^2$. The vacuum equations (3.2) are just a particular case of (3.9) or equivalently (3.10). They emerge when we add the requirement of time-independence and vanishing of the chromo-electric field E_i. From (3.8) it is manifest that all our vacua (3.3) possess a vanishing action and they are all equivalent from an energetic point of view.

It is natural to ask now what is the meaning of the Euclidean time-dependent solutions of (3.9). The action on these solutions reduces to

$$S_{\text{class}} = \mp\frac{1}{\mu}\int_{S^2} d\theta d\varphi \sin\theta \int_{-\infty}^{\infty} dt\partial_t \mathrm{Tr}(\phi_3^2)\,, \tag{3.11}$$

which is finite, and thus relevant for a semiclassical analysis of the theory, if and only if $\phi_3(t=-\infty) = \frac{\mathfrak{f}_{-\infty}}{2\mu R^2}$ and $\phi_3(t=\infty) = \frac{\mathfrak{f}_\infty}{2\mu R^2}$. In other words, these solutions are interesting if and only if they interpolate between two vacua: one at $t = -\infty$ and the other at $t = +\infty$. Their finite action is then given by

$$S_{\text{class}} = \mp\frac{1}{\mu}\int_{S^2} \sin\theta d\theta d\varphi \int_{-\infty}^{\infty} dt\partial_t \mathrm{Tr}(\phi_3^2) = \mp\frac{\pi}{g_{YM}^2 R}(\mathrm{Tr}(\mathfrak{f}_\infty^2) - \mathrm{Tr}(\mathfrak{f}_{-\infty}^2))\,, \tag{3.12}$$

where we have reintroduced the relevant coupling constant factors. We recognize the characteristics of *instantons* in these (Euclidean) time-dependent solutions. At the quantum level, they will possibly induce a tunneling process between the different vacua. At zero temperature Lin [45] discussed the effect of these instantons from the gauge theoretical side, at weak coupling, and from the gravity side, that should describe the strong-coupling limit of the theory (see also [46]), finding precise agreement in both regimes. Moreover he argued, in analogy with the plane-wave matrix model, that because of the presence of fermionic zero-modes[4] around these instanton solutions, the path-integral for the tunneling amplitude is zero. The vacuum energies would not be corrected and the vacua are exactly protected at the quantum mechanical level: in particular they should remain degenerate. This kind of instantons has also been recently considered in [47].

In the rest of the paper, in any case, we shall neglect the effect of these solutions since we shall work at zero-coupling and in this limit the probability of tunneling is exponentially suppressed anyway.

4. Free SYM partition functions in monopole vacua

In this section we shall derive the finite temperature partition function in the BPS vacua (3.3), taking the limit $g^2_{YM}R \to 0$. We follow a path-integral approach where the computation is reduced to the evaluation of one-loop functional determinants in the monopole backgrounds. Since at finite temperature the Euclidean time is a circle S^1 of length $\beta = 1/T$, we can also allow for a flat-connection a wrapping this S^1. The mode a will play a very special role because it is the only zero-mode in the decomposition into Kaluza-Klein modes on $S^2 \times S^1$. Consequently, as stressed in [9], the fluctuations described by a are always strongly coupled, including in the limit $g^2_{YM}R \to 0$.

When the vacuum is trivial, there is no breaking of the $\mathrm{U}(N)$ gauge symmetry and the final result for the partition function is given by a matrix integral over the unitary matrix $U = \exp\big[i\beta a\big]$

$$\mathcal{Z}(\beta) = \int [dU] \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n}\left[z_B(x^n) + (-1)^{n+1} z_F(x^n)\right] \mathrm{Tr}(U^n)\mathrm{Tr}(U^{-n})\right\}. \tag{4.1}$$

The functions $z_{B,F}(x)$ are respectively the bosonic and fermionic single-particle partition functions (here $x = e^{-\beta}$), counting the one-particle states of the theory without the degeneracy coming from the dimension of the representation (the adjoint representation Adj in our case) and without any gauge invariant constraint

$$z_{B,F}(x) = \sum_i e^{-\beta E_i^{(B,F)}}. \tag{4.2}$$

The explicit form of the thermal partition function is obtained by integrating over the matrix U [8, 9]

$$\begin{aligned}\mathcal{Z}(\beta) = &\sum_{n_1=0}^{\infty} x^{n_1 E_1^B} \sum_{n_2=0}^{\infty} x^{n_2 E_2^B} \ldots \sum_{m_1=0}^{\infty} x^{m_1 E_1^F} \sum_{m_2=0}^{\infty} x^{m_2 E_2^F} \ldots \times \\ &\#\,\text{of singlets in } \{\mathrm{sym}^{n_1}(Adj) \otimes \mathrm{sym}^{n_2}(Adj) \otimes \cdots \\ &\otimes \mathrm{antisym}^{m_1}(Adj) \otimes \mathrm{antisym}^{m_2}(Adj) \otimes \cdots\} : \end{aligned} \tag{4.3}$$

[4]The instantons are 1/2 BPS solutions and therefore we expect 8 fermionic zero-modes associated to the broken supersymmetries

the partition function is expressed as a sum over the occupation numbers of all modes, with a Boltzmann factor corresponding to the total energy, and a numerical factor that counts the number of singlets in the corresponding product of representations. Particle statistics requires to symmetrize (antysimmetrize) the representations corresponding to identical bosonic (fermionic) modes.

The same result can also be obtained starting from

$$\mathcal{Z}(\beta) = \mathrm{Tr}\left[e^{-\beta H}\right] \equiv \mathrm{Tr}\left[x^{H}\right], \tag{4.4}$$

where H is the Hamiltonian of the theory. To calculate (4.4) at zero coupling we need a complete basis of states of the free theory or, thanks to the state-operator correspondence, of gauge-invariant operators and we should count them weighted by x to the power of their energy. A complete basis for arbitrary gauge-invariant operators follows naturally after we specify a complete basis of single-trace operators. At the end, one can write (4.4) in terms of single-particle partition functions $z^{R}_{B,F}(x)$ [9] as

$$\mathcal{Z}(\beta) = \int [dU] \exp\left\{\sum_{R}\sum_{n=1}^{\infty}\frac{1}{n}\left[z^{R}_{B}(x^{n}) + (-1)^{n+1} z^{R}_{F}(x^{n})\right]\chi_{R}(U^{n})\right\}, \tag{4.5}$$

where the sum is taken over the representations R of the $\mathrm{U}(N)$ gauge group[5] and $\chi_R(U)$ is the character for the representation R. The result (4.1) is reproduced when all fields are in the adjoint representation: the variable U has to be identified as the holonomy matrix along the thermal circle, *i.e.* the Polyakov loop. The path-integral approach provides therefore a physical interpretation for the unitary matrix U, otherwise missing in the Hamiltonian formalism. On the other hand the Hamiltonian construction explains how the group integration forces the projection into color singlets and how it emerges the structure of the full Hilbert space.

From the previous results we learn that once the representation content is specified, the full partition function is completely encoded into the single-particle partition functions $z^{R}_{B,F}$. However, the structure of the gauge group is more complicated on monopole backgrounds, consisting into a direct product of $\mathrm{U}(N_I)$ factors: consequently our constituents fields transform also under bifundamental representations, producing additional complications for the explicit expression of the matrix model. We also remark that bifundamental fields can transform non-trivially under $\mathrm{U}(1)$ rotations and implementing the Gauss's law hides some subtleties in three dimensions, when background monopole fluxes are present [25]: this potential additional freedom could affect non-trivially the spectrum of physical operators in our theory. For the theory we are investigating, however, the free-field spectrum is simply obtained by truncating the four-dimensional parent theory, suggesting that the $\mathcal{N}=8$ counting is conveniently performed through the relevant $\mathrm{U}(1)$ projection on the $\mathcal{N}=4$ single-particle partition functions. This is what we do in appendix C, where we construct the projector that eliminates all the fields which are not invariant under the $\mathrm{U}(1)$ and we derive, even in the non-trivial vacuum, the single-particle partition functions for

[5] We consider the possibility to have fields in an arbitrary representation.

bosons and fermions. While this is certainly the quickest way to obtain these quantities, we prefer to adopt here a path integral approach which in turn provides also the contributions of fermions and bosons to the Casimir energy and allows for a careful treatment of the fermion zero modes. In the path-integral computation all the subtleties will be treated in the well-defined framework of the ζ-function regularization procedure and in this section we present only the final results, referring for the technical details to appendix B.

4.1 Scalars

Let us first describe the contribution of the six $\mathrm{SU}(4)_R$ scalars ϕ_{ij} to the partition function in the background (3.3) and in presence of the flat-connection a: it amounts to the evaluation of the determinant of the scalar kinetic operator. We have to solve the associated eigenvalue problem, i.e.

$$-\hat{\Box}\phi_{ij} + \frac{\mu^2}{4}\phi_{ij} + [\hat{\phi}_3, [\hat{\phi}_3, \phi_{ij}]] = \lambda\phi_{ij}, \tag{4.6}$$

where the hatted quantities are computed in the relevant background. In the following we shall drop the subscript $_{ij}$ and we shall consider just one field denoted by ϕ. The total result at the level of free energy is then obtained by multiplying by six the single-component contributions. Since ϕ is a matrix-valued field, we shall expand it in the Weyl-basis, whose elements are the generators H_i of the Cartan subalgebra and the ladder operators E^α

$$\phi = \sum_{i=1}^{N-1} \phi_i H^i + \sum_{\alpha\in\mathrm{roots}} \phi_\alpha E^\alpha. \tag{4.7}$$

We shall also expand the background fields in this basis and define the following two accessory quantities

$$a_\alpha = \langle\alpha|a\rangle \quad \text{and} \quad q_\alpha = \frac{\langle\alpha|\mathfrak{f}\rangle}{2}. \tag{4.8}$$

Here a_α denotes the projection of the flat-connection a along the root α and q_α is the effective monopole charge measured along the same root. Once the time-dependence is factored out, the original eigenvalue problem splits into two subfamilies: $N(N-1)$ independent eigenvalues coming from each direction along the ladder generator and $N-1$ independent eigenvalues coming from the directions along the Cartan subalgebra. We can simply focus our attention on the first family, since the latter can be obtained as a limiting case for $a_\alpha,\ q_\alpha \to 0$. The relevant eigenvalue equation can be solved algebraically if we introduce the angular momentum operator in the presence of a $\mathrm{U}(1)$ monopole of charge q_α, as explained in appendix B, and the resulting spectrum does not depend on the sign of q_α. By using ζ-function regularization, the scalar contribution to the effective action can be easily computed as

$$\Gamma^{Sc.} = \sum_{\alpha\in\mathrm{roots}} \left(\frac{|q_\alpha|}{12}\left(4|q_\alpha^2|-1\right)\beta\mu + \sum_{n=1}^{\infty} \frac{z_{q_\alpha}^{scal.}(x^n)}{n} e^{in\beta a_\alpha} \right) + (N-1)\sum_{n=1}^{\infty} \frac{z_0^{scal.}(x^n)}{n}, \tag{4.9}$$

where the scalar single-particle partition function is given by

$$z_{q_\alpha}^{scal.}(x) = x^{|q_\alpha|+1/2}\left(\frac{1+x}{(1-x)^2} + \frac{2|q_\alpha|}{1-x}\right). \tag{4.10}$$

4.2 Vectors

Evaluating the contribution of the system (A_μ, ϕ_3) is more subtle and involved: the fields are coupled through the Chern-Simons term and the Lagrangian for A_μ requires a gauge-fixing procedure, with the consequent addition of a ghost sector. A convenient choice for such a gauge-fixing appears to be

$$\mathcal{L}_{g.f.} = (\hat{D}_\nu A^\nu - i[\hat{\phi}_3, \phi_3])^2, \tag{4.11}$$

where $\hat{\phi}_3 = \frac{\mu \mathfrak{f}}{2}$ and the hatted derivative is defined in (B.19). With this choice some of the mixing-terms in the Euclidean quadratic Lagrangian cancel and we obtain the relevant eigenvalue-problem for computing the vector-scalar contribution to the partition function: it is defined by the system of coupled equations, written explicitly in (B.26). Since both the geometrical and the gauge background are static, the time-component of the vector field A_0 decouples completely from the eigenvalue system and satisfies the massless version of the scalar equation previously studied. For the moment we shall forget about A_0 since its contribution will be cancelled by the ghost determinant. We are left with a purely two-dimensional system where all the indices run only over space: the spectrum is again conveniently determined by factoring out the time-dependence and projecting the eigenvalue equations on the Weyl basis. We remark that the equations involve also the Laplacian on vectors in the background of a monopole of charge q_α, besides the Laplacian on scalars. The full computation of the spectrum is reported in appendix B: we obtained three families of eigenvalues, denoted by λ_+, λ_- and λ_3. The contribution of λ_3 will be cancelled by the ghost determinant and we just consider, at the moment, the first two families $\lambda_\pm$, which instead yield the actual vector determinant in the roots sector

$$\Gamma_r^V = \sum_{\alpha \in \text{roots}} \left(-\frac{1}{3} \left(4q_\alpha^3 + 5q_\alpha\right) \beta\mu - 2 \sum_{n=1}^{\infty} \frac{z_{q_\alpha}^{vec.}(x^n)}{n} e^{in\beta a_\alpha} \right), \tag{4.12}$$

where

$$z_{q_\alpha}^{vec.}(x) = x^{q_\alpha} \left[\frac{4x}{(1-x)^2} - 1 + 2q_\alpha \frac{1+x}{1-x} \right]. \tag{4.13}$$

We remark that the results (4.12) and (4.13) were shown to hold under the initial assumption $q_\alpha \geq 1$. The extra-cases to be considered are $q_\alpha = \frac{1}{2}, 0$. By recomputing the spectrum for $q_\alpha = 1/2$ we get the same results: quite surprisingly this does not happen, instead, for $q_\alpha = 0$ and we get

$$\Gamma_r^V(q_\alpha = 0) = -2 \sum_{n=1}^{\infty} \frac{z_0^{vec.}(x^n)}{n} e^{in\beta a_\alpha} \quad \text{with} \quad z_0^{\text{vec}}(x) = \frac{4x}{(1-x)^2}, \tag{4.14}$$

a factor -1 missing in the limit. To complete the discussion, we notice that, when multiplied by $(N-1)$, (4.14) is the contribution of the Cartan components; the results (4.12) and (4.13) extends also to negative charges q_α by simply replacing q_α with $|q_\alpha|$.

4.3 Ghosts and A_0

Let us discuss now the contributions to the partition function of the eigenvalues λ_3, of the field A_0 and of the determinant of ghost operator

$$-\hat{\Box}\cdot+[\hat{\phi}_3,[\hat{\phi}_3,\cdot]]: \tag{4.15}$$

they do not cancel completely but, importantly, they give a measure of integration for the flat-connection. It is possible to show that when $q_\alpha \neq 0$ we have a complete cancellation of the different contributions: crucially for $q_\alpha = 0$ this does not happen and a modification of the measure for the flat-connection is induced

$$\prod_{\substack{\alpha\in\text{roots}\\ \text{with } q_\alpha=0}} 2ie^{-i\frac{\beta a_\alpha}{2}}\sin\left(\frac{\beta a_\alpha}{2}\right)=\prod_{\substack{\alpha\in\text{positive roots}\\ \text{with } q_\alpha=0}} 4\sin^2\left(\frac{\beta a_\alpha}{2}\right). \tag{4.16}$$

The meaning of this measure is quite transparent: the monopole background breaks the original $\mathrm{U}(N)$ invariance to the subgroup $\prod_{I=1}^{k}\mathrm{U}(N_I)$, (4.16) being the product of the Haar measure of each $\mathrm{U}(N_I)$ component, as can be easily checked by recalling the explicit form of the roots and the definition of q_α. As a matter of fact, in non-trivial monopole backgrounds, when we shall write the integral over the flat-connections we will be naturally led to consider a unitary *multi-matrix* model instead of an ordinary one.

4.4 Fermions

The contribution of the fermions to the total partition function needs a careful analysis. At first sight, apart from having antiperiodic boundary conditions along the time circle, the computation of the fermion determinants seems to follow closely the bosonic cases. We have again $N(N-1)$ independent eigenvalues coming from each direction along the ladder generators and $N-1$ independent eigenvalues coming from the directions along the Cartan subalgebra, that can obtained as limit of vanishing flux. The computation of the spectrum is quite technical as in the vector case and boils down in solving the eigenvalue problem for a family of effective massless Dirac operators $\mathfrak{D}^{(\alpha)}$ (see app. B.4) on the two-sphere, in the effective monopole backgrounds provided by q_α. The spectrum of $\mathfrak{D}^{(\alpha)}$, as expected in two dimensions, consists in a set non-vanishing eigenvalues, symmetric with respect the zero, and in a finite kernel, as predicted by the Atiyah-Singer theorem. These zero-modes are chiral and can be classified by using the eigenvalues of the operator $(\sigma\cdot\hat{r})$, playing the role of γ_5: we shall denote $\nu_\pm$ the number of zero modes with eigenvalue ± 1. A simple application of the index theorem shows that $\nu_+=|q_\alpha|-q_\alpha$ and $\nu_-=|q_\alpha|+q_\alpha$, namely for positive q_α we have only zero modes with negative chirality and viceversa. As shown in appendix B.4, the contribution of the first set of eigenvalues to the effective action can be easily evaluated

$$\Gamma_1^S=\sum_{\alpha\in\text{roots}}\left(-\frac{\beta\mu}{3}\left(2|q_\alpha|^3+3|q_\alpha|^2+|q_\alpha|\right)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^{spin.}_{q_\alpha 1}(x^n)e^{i\beta n a_\alpha}\right), \tag{4.17}$$

with

$$z^{spin.}_{q_\alpha 1}(x)=2x^{|q_\alpha|+1}\left(\frac{1}{(1-x)^2}+\frac{|q_\alpha|}{1-x}\right)\left(x^{\frac{1}{4}}+x^{-\frac{1}{4}}\right). \tag{4.18}$$

Next we consider the contribution of the zero-modes of the effective Dirac operators: in a monopole background, this subsector originates the spectral asymmetry [48] of the three dimensional fermionic operator and therefore the potential appearance of a parity violating part in the effective action. In particular, we could expect the generation of the Chern-Simons anomalous term (we refer to [33, 34] for a complete discussion of this issue). Concretely, in our case, the explicit computation of the zero-mode contribution amounts to evaluate a family of one-dimensional massive fermion determinants, in a flat-connection background (see appendix B.4). It is well-known that the ζ-function regularization scheme carries an intrinsic regularization ambiguity[6] in this case, depending on the choice of some branch-cuts in the s-plane, affecting the local terms in the effective action [33, 34]. For us all the different possibilities boil down to two alternatives: we can regularize the contributions associated to the *zero-modes* of negative and positive chirality by choosing opposite cuts in defining the complex power of the eigenvalues (one on the real positive axis and the other on the real negative axis) or by choosing the same cut. We find quite natural to use the same procedure for *all* the four fermions present in the theory: we surely preserve the R-symmetry and the global non-abelian symmetry in this way. Within this choice, the following results hold from our one-dimensional fermion determinants: taking opposite cuts we get

$$\Gamma^S_{0,A} = \sum_{\alpha\in\text{roots}} (1-r)\beta\mu\left(q^2_\alpha + \frac{|q_\alpha|}{4}\right) - \sum_{\alpha\in\text{roots}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}2|q_\alpha|x^{n|q_\alpha|}e^{i\beta n a_\alpha}x^{\frac{n}{4}}. \tag{4.19}$$

Here $r=\pm1$ and its specific value depends on the cut selected for the zero-modes of positive chirality. Choosing instead the same cuts we obtain

$$\Gamma^S_{0,B} = \sum_{\alpha\in\text{roots}}\left[\beta\mu\left(|q_\alpha|^2 + \frac{|q_\alpha|}{4}\right) + ir\beta a_\alpha q_\alpha\right] - \sum_{\alpha\in\text{roots}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}2|q_\alpha|x^{n|q_\alpha|}e^{i\beta n a_\alpha}x^{\frac{n}{4}}. \tag{4.20}$$

Again $r=\pm1$ according to the specific choice of the cut (real positive or negative axis): we must stress, however, that this ambiguity will become irrelevant when we shall perform the integration over the flat-connections.

We remark that there is an important difference between the two expressions: in the second case we have a new term in the effective action, depending explicitly on the flat connection. To understand its nature, it can be equivalently written as

$$ir\sum_\alpha \beta q_\alpha a_\alpha = ir\beta(N\mathrm{Tr}(a\mathfrak{f}) - \mathrm{Tr}(a)\mathrm{Tr}(\mathfrak{f})). \tag{4.21}$$

We immediately recognize the SU(N) part of the usual Chern-Simons term, calculated in our particular background. The related regularization choice is therefore consistent with the intrinsic parity anomaly of three dimensional gauge theories. We stress that the above contribution arises just in the monopole vacua and it is related to non-perturbative

[6] This ambiguity is not something peculiar of the ζ-function regularization, but it appears in different forms also in other regularizations: in the usual Pauli-Villars approach, for example, this ambiguity translates into a dependence of the local terms in the effective action on the sign of the mass of the regulator.

properties of the fermion determinants. We also observe that the two results differ in the charge-dependent contribution linear in β, and we will see this to modify crucially the Casimir energy.

Summing now, in both cases, the kernel contribution to Γ^S_1 we get

$$\Gamma^S_A = \sum_{\alpha\in\mathrm{roots}}\left(-\frac{\beta\mu}{12}\left(8|q_\alpha|^3+12r|q_\alpha|^2+(3r+1)|q_\alpha|\right)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^{spin.}_{q_\alpha}(x^n)e^{i\beta n a_\alpha}\right), \tag{4.22}$$

with the first choice and

$$\Gamma^S_B = \sum_{\alpha\in\mathrm{roots}}\left(-\frac{\beta\mu}{3}\left(2|q_\alpha|^3+\frac{|q_\alpha|}{4}\right)+irq_\alpha a_\alpha-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^{spin.}_{q_\alpha}(x^n)e^{i\beta n a_\alpha}\right), \tag{4.23}$$

in the latter. Happily the single-particle partition function is the same for both the regularization choices

$$z^{spin.}_{q_\alpha}(x) = x^{|q_\alpha|}\left(\frac{2x}{(1-x)^2}+\frac{2|q_\alpha|\sqrt{x}}{1-x}\right)\left(x^{\frac{1}{4}}+x^{-\frac{1}{4}}\right). \tag{4.24}$$

The contribution of the Cartan components is of course obtained from the above results by simply setting $q_\alpha = 0$.

4.5 Partition functions

The next step is to collect the different contributions, coming from the functional determinants, and write down the total result as a compact integral over unitary matrices. According to the previous discussion, we must distinguish two cases, depending on the form of the spinor determinant (4.22) or (4.23). We shall first consider the choice (4.22). The complete effective action, obtained by including roots and Cartan contributions with the appropriate multiplicities, can be expressed as

$$\begin{aligned} S_{eff.} =& -\beta V_0 + \sum_{\alpha\in\mathrm{roots}}\sum_{n=1}^{\infty}\frac{1}{n}(6z^{scal.}_{q_\alpha}(x^n)+z^{vec.}_{q_\alpha}(x^n)+(-1)^{n+1}4z^{spin.}_{q_\alpha}(x^n))e^{in\beta a_\alpha}+\\ &+(N-1)\sum_{n=1}^{\infty}\frac{1}{n}(6z^{scal.}_0(x^n)+z^{vec.}_0(x^n)+(-1)^{n+1}4z^{spin.}_0(x^n))\equiv\\ \equiv& -\beta V_0+\sum_{\alpha\in\mathrm{roots}}\sum_{n=1}^{\infty}\frac{1}{n}z^{tot.}_{q_\alpha}(x^n)e^{in\beta a_\alpha}+(N-1)\sum_{n=1}^{\infty}\frac{1}{n}z^{tot.}_0(x^n), \end{aligned} \tag{4.25}$$

where we have introduced the total single-particle partition functions and the Casimir energy V_0 of the configuration

$$V_0 = r\sum_{\alpha\in\mathrm{roots}}(4|q_\alpha|^2+|q_\alpha|). \tag{4.26}$$

The matrix structure hidden in (4.25) appears manifest when writing the original Polyakov loop $U=\exp(i\beta a)$, associated to the diagonal flat-connection a, through k sub-matrices U_I acting on the invariant subspaces implicitly defined by the monopole background (3.4).

The $N_I \times N_I$ unitary matrices U_I have the form $U_I = \mathrm{diag}(e^{i\beta a^I_1}, \ldots, e^{i\beta a^I_{N_I}})$, where we have parameterized the original flat connection a as follows:

$$a = \mathrm{diag}(\underbrace{a^1_1, \ldots, a^1_{N_1}}_{N_1}, \underbrace{a^2_1, \ldots, a^2_{N_2}}_{N_2}, \ldots\ldots, \underbrace{a^I_1, \ldots, a^I_{N_I}}_{N_I}, \cdots). \tag{4.27}$$

Let us consider now the subset $\mathcal{A}_{IJ}$ of the positive roots[7] of SU(N) whose first and second non vanishing entries belong respectively to the I^{th} and J^{th} invariant subspace of $\mathfrak{f}$. The effective charges $q_\alpha = \frac{\langle \alpha|\mathfrak{f}\rangle}{2} = \frac{n_I - n_J}{2}$ and, consequently, the $z^{tot.}_{q_\alpha}$ take always the same value for this class of roots. The sum over roots on this subset reduces to

$$\sum_{\alpha\in\mathcal{A}_{IJ}} e^{in\beta a_\alpha} = \sum_{i=1}^{N_I}\sum_{j=1}^{N_J} e^{in\beta(a^I_i - a^J_j)} = \mathrm{Tr}(U^n_I)\mathrm{Tr}(U^{\dagger n}_J); \tag{4.28}$$

the analogous subsector $\bar{\mathcal{A}}_{IJ}$ given by the negative roots yields $\mathrm{Tr}(U^{\dagger n}_I)\mathrm{Tr}(U^n_J)$. We remark that the pre-factor $z^{tot.}_{q_\alpha}$ is however the same for both cases since it depends just on the modulus of the effective monopole charge. The subset of roots $\mathcal{B}_I$ whose first and second non vanishing entries live in the same I^{th} invariant subspace of $\mathfrak{f}$ have instead effective monopole charge zero. Then the contribution of this subsector is simply given by

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n} z^{tot.}_0(x^n) \sum_{\alpha\in\mathcal{B}_I} e^{in\beta a_\alpha} &= \sum_{n=1}^{\infty}\frac{1}{n} z^{tot.}_0(x^n) \sum_{i\neq j=1}^{N_I} e^{in\beta(a^I_i - a^I_j)} = \\ &= \sum_{n=1}^{\infty}\frac{z^{tot.}_0(x^n)}{n}(\mathrm{Tr}(U^{\dagger n}_I)\mathrm{Tr}(U^n_I) - N_I). \end{aligned} \tag{4.29}$$

Because of the results (4.28) and (4.29), it is convenient to change our notation and to define the $k \times k$ matrix-valued single-particle partition function $z^{tot.}_{IJ}$: the diagonal elements are $z^{tot.}_{II} = z^{tot.}_0$, the off-diagonal ones are instead identified with the function $z^{tot.}_{q_\alpha}$, associated to the charge $\frac{n_I - n_J}{2}$. The matrix $z^{tot.}_{IJ}$ is symmetric since everything depends just on the modulus of the charge. The complete effective acton takes the elegant form

$$S_{eff.} = -\beta V_0 + \sum_{IJ}\sum_{n=1}^{\infty}\frac{1}{n} z^{tot.}_{IJ}(x^n)\mathrm{Tr}(U^n_I)\mathrm{Tr}(U^{\dagger n}_J) - \sum_{n=1}^{\infty}\frac{1}{n} z^{tot.}_{II}(x^n). \tag{4.30}$$

The last term drops if we consider U(N) instead of SU(N). Remarkably the structure of the matrix action is perfectly consistent with the measure found in (4.16), which is exactly the Haar measure for this multi-matrix model.

The above analysis is practically unaltered when considering the fermionic contribution (4.23) in the effective action, except on a couple of points. It changes the value of the Casimir energy V_0, which now vanishes identically, and we have a new important addition to (4.30), that can expressed in terms of the determinants of the unitary matrices U_I

$$ir\beta \sum_{\alpha\in\mathrm{roots}} q_\alpha a_\alpha = \log\left(\prod_{I=1}^{k}\det(U_I)^{r(Nn_I - Q)}\right) = r\sum_{I=1}^{k}(Nn_I - Q)\log(\det(U_I)), \tag{4.31}$$

[7]The roots of SU(N) are all the $N(N-1)$ permutations of the $N-$vector $(1, -1, 0, \cdots, 0)$ and they can be separated in *positive* and *negative* according to the sign of the first non zero entry.

where $Q = \sum_{I=1}^{k} N_I n_I$. As a first remark, we notice that new contributions depends still on the differences $n_I - n_J$, consistently with the decoupling of the total $\mathrm{U}(1)$ charge of the monopole configuration. Then we observe that the two different values $r = \pm 1$, related to our regularization choice, produce the same result when integrating over the unitary group: the difference can be reabsorbed just changing integration variable $U_I \mapsto (U_I)^{-1}$, which leaves the measure and (4.30) unaltered. From now on, we shall set $r = 1$.

In the trivial vacuum we obtain a partition function that is a straightforward generalization of the unitary matrix model discussed in [9]

$$\mathcal{Z} = \int dU \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} z_0^{tot.}(x^n)\mathrm{Tr}(U^n)\mathrm{Tr}(U^{\dagger n})\right) \tag{4.32}$$

where the function $z_0^{tot.}(x^n)$ encodes the dynamical content of the three-dimensional supersymmetric theory. Notice that the Casimir energy is identically zero, since it vanishes for each contribution both bosonic and fermionic.

The situation changes in non-trivial monopole vacua: we get respectively

$$\mathcal{Z}_A = \int \prod_{I=1}^{k} [dU_I] \exp\left(-\beta V_0 + \sum_{IJ}\sum_{n=1}^{\infty} \frac{1}{n} z_{IJ}^{tot.}(x^n)\mathrm{Tr}(U_I^n)\mathrm{Tr}(U_J^{\dagger n})\right) \tag{4.33}$$

and

$$\mathcal{Z}_B = \int \prod_{I=1}^{k} [dU_I] \exp\left(\sum_{IJ}\sum_{n=1}^{\infty} \frac{1}{n} z_{IJ}^{tot.}(x^n)\mathrm{Tr}(U_I^n)\mathrm{Tr}(U_J^{\dagger n})\right) \prod_{I=1}^{k} \det(U_I)^{(N n_I - Q)}, \tag{4.34}$$

depending on our regularization choice. First of all we see that the partition function is related to a unitary multi-matrix model: the gauge group is broken in factors and states in the bifundamental representation are present, with energies clearly encoded into the off-diagonal entries of the single-particle partition function $z_{IJ}^{tot.}$. Let us discuss on general grounds the effects of the different choices for the fermion determinants. A first mild diversity arises in the Casimir energies: from (4.22) we have a non-vanishing V_0, with arbitrary sign, while (4.23) leads to a vanishing result. We recall that the Casimir energy is supposed to correspond to the mass of the dual geometry [39]: in the first case it seems that different backgrounds supports different, monopole dependent, masses, suggesting a possible lifting of the vacua degeneracy at quantum level. The second choice is instead consistent with the believed degeneracy: unfortunately no computation from the gravitational side seems to be available up to now and we do not have further insights on the meaning of the different results.

The presence of the new terms (4.31) in the matrix model (4.34) can be, instead, better understood at the level of partition functions. First of all we notice that the matrix integral implementing the Gauss's law is actually over *unitary* matrices U_I: the $\mathrm{U}(1)$ phases contained into the the U_I's play a non-trivial role in the monopole background. This has to be contrasted with the trivial vacuum: there the effective action is invariant under $\mathrm{U}(1)$ transformations and we can simply forget the integration over the center. In the non-trivial vacuum the resulting effective action (4.30) is not invariant under phase rotations, as an effect of the off-diagonal terms in the single-particle partition function, and the U(1) integrations precisely correspond to selection rules in the bifundamental sector. It is not difficult to realize that within the first regularization the matrix integrals select states having vanishing U(1) charge, with respect to all $\mathrm{U}(N_I)$ group factors. To understand the effect of the new terms in (4.34) instead, we simply observe that the determinants depend just on the U(1) phases and modify non-trivially the selection rules of the bifundamental sectors, according to the charges of the monopole background. We shall say in this case that our regularization procedure correspond to the choice of a charged vacuum, as discussed in [26], while we will refer to the first possibility as to the uncharged vacuum. Since at the quantum field theory level both choices seems to be allowed, we think it is instructive to investigate the thermodynamics in both cases, deferring a deeper understanding of the different possibilities to future studies, in the context of supersymmetry and gravitational duals.

We end this section introducing the simple modification to the effective action due to chemical potentials for the SU(4) R-charge. In the path integral approach their effect amounts to simply adding an imaginary SU(4) flat connection $\mathbb{A}^{\mathbf{R}} = i(\Omega_1 Q_1^{\mathbf{R}} + \Omega_2 Q_2^{\mathbf{R}} + \Omega_3 Q_3^{\mathbf{R}})$ in the Euclidean time direction. Here $Q_i^{\mathbf{R}}$ are the Cartan generators of SU(4) and $\mathbf{R}$ denotes the relevant representation: **4** for the spinors and **6** for the scalars. One finds the new partition functions

$$\begin{aligned} 4z^{spin.}_{q_\alpha} \mapsto z^{spin.}_{IJ} &= x^{|q_\alpha|}\left(\frac{2x}{(1-x)^2}\sum_{p=1}^{4}\left(x^{\frac{1}{4}}y^{-\widetilde{\Omega}_p}+x^{-\frac{1}{4}}y^{\widetilde{\Omega}_p}\right)+\right.\\ &\qquad\left. +2|q_\alpha|\frac{x^{\frac{1}{4}}}{1-x}\sum_{p=1}^{4}\left(y^{-\widetilde{\Omega}_p}+x^{\frac{1}{2}}y^{\widetilde{\Omega}_p}\right)\right), \qquad (4.35)\\ 6z^{scal.}_{q_\alpha} \mapsto z^{scal.}_{IJ} &= x^{|q_\alpha|+1/2}\left(\frac{x+1}{(1-x)^2}+2|q_\alpha|\frac{1}{1-x}\right)\sum_{p=1}^{3}\left(y^{\Omega_p}+y^{-\Omega_p}\right), \end{aligned}$$

with $y = e^{-\beta}$ and

$$\begin{aligned} \tilde{\Omega}_1 &= \frac{1}{2}(\Omega_1+\Omega_2+\Omega_3) \qquad \tilde{\Omega}_2 = \frac{1}{2}(\Omega_1-\Omega_2-\Omega_3)\\ \tilde{\Omega}_3 &= \frac{1}{2}(-\Omega_1+\Omega_2-\Omega_3) \qquad \tilde{\Omega}_4 = \frac{1}{2}(-\Omega_1-\Omega_2+\Omega_3). \end{aligned} \qquad (4.36)$$

5. Thermodynamics in the trivial vacuum

We have seen in the previous section that the thermodynamics in the trivial vacuum is governed, in the zero-coupling approximation, by the *one-component* unitary matrix model

$$\mathcal{Z} = \int dU \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}z_0^{tot.}(x^n)\mathrm{Tr}(U^n)\mathrm{Tr}(U^{\dagger n})\right) \qquad (5.1)$$

where the function $z_0^{tot.}(x^n)$ encodes the dynamical content of the three-dimensional supersymmetric theory. Notice that the Casimir energy is identically zero, since it vanishes for each contribution both bosonic and fermionic.

When N is large we can trade the integration in (5.1) over the unitary group for an integration over the normalized distribution function $\rho(\theta)$ of the continuous eigenvalues $e^{i\theta}$ of U, with $-\pi < \theta \leq \pi$. More precisely we can write the integral over the unitary matrices in terms of the Fourier-modes $(\rho_n, \bar{\rho}_n)$ defined as

$$\rho(\theta) = \frac{1}{2\pi} + \sum_{n=1}^{\infty}(\rho_n e^{in\theta} + \bar{\rho}_n e^{-in\theta}) \ . \tag{5.2}$$

Following [8, 9], we can then reduce the integral to the standard form

$$\mathcal{Z} = \int D\rho_n D\bar{\rho}_n \exp\left(-N^2 \sum_{n=1}^{\infty} \rho_n \bar{\rho}_n V(x^n)\right) \qquad \text{with} \quad V(x^n) = \frac{1}{n}(1 - z_0^{tot.}(x^n)) \ . \tag{5.3}$$

In the large N limit, (5.3) is dominated by the absolute minimum of the quadratic action $S = \sum \rho_n \bar{\rho}_n V(x^n)$ which is reached for $\rho_n = 0$ for every n if $V(x^n)$ is positive definite. For small temperatures, namely small x, the function $V(x^n)$ is positive for any n and close to $1/n$ since $V(x^n) \sim \frac{1}{n}$ for $x \ll 1$. (Recall that $z_0^{tot.}(x^n)$ vanishes as x approaches zero.) Therefore the partition function is 1 at the leading order and it is simply given by the small fluctuation around the minimum at the subleading order:

$$\mathcal{Z} \propto \prod_{n=1}^{\infty} \frac{1}{(1 - z_0^{tot.}(x^n))}. \tag{5.4}$$

When we increase the temperature, x approaches 1 and the above description is reliable up to the smallest value x_c where $V(x^n)$ becomes negative. Since $z_0^{tot.}(x)$ is a monotonic function ranging from 0 to infinity, this value always exists and it is reached for $n = 1$, namely

$$V(x_c) = 1 - z_0^{\text{tot}}(x_c) = 0. \tag{5.5}$$

This algebraic condition, whose explicit form is

$$\begin{aligned} V(x_c) = 1 - z_0^{tot.} &= 1 - (4z_0^{spin.} + 6z_0^{scal.} + z_0^{vec.}) = \\ &= \frac{\left(\sqrt[4]{x_c} + 1\right)^4 \left(x_c - 4{x_c}^{3/4} + 4\sqrt{x_c} - 4\sqrt[4]{x_c} + 1\right)}{(1 - x_c)^2} = 0, \end{aligned} \tag{5.6}$$

can be exactly solved, since it can be reduced to an equation of fourth degree. It possesses just one solution in the interval $[0, 1]$ given by

$$x_c = \left(2 + 2\sqrt{2} - \sqrt{11 + 8\sqrt{2}}\right)^2 \simeq (0.104688)^2. \tag{5.7}$$

It is interesting to compare this value with the critical temperature computed in [39] for $\mathcal{N} = 4$ on $S^3/\mathbb{Z}_k$. This theory should in fact reproduce our model when k goes to infinity. However, the three-dimensional theory obtained in this limit lives on a S^2 sphere whose radius is half of the radius of the original S^3: this means that $x_c = \lim_{k\to\infty} x_c^2(k)$. To facilitate the comparison with the four dimensional literature and in particular with the results of [20, 39] in what follows we shall replace the basic variable x with x^2. In [39] the $x_c(k)$ for $k = 10$ is 0.104689 which is already very close to (5.7).

Above this critical value the integral (5.3) is no longer dominated by the trivial minimum $\rho_n = \bar{\rho}_n = 0$ and one has to look for other saddle-points [8, 9]. Following [9], one can easily show that above x_c the dynamics is governed by a distribution different from zero only in the interval $[-\theta_0, \theta_0]$ and given, in first approximation,[8] by

$$\rho(\theta) = \frac{\cos\left(\frac{\theta}{2}\right)}{\pi \sin^2\left(\frac{\theta_0}{2}\right)} \sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} \quad \text{with} \quad \cos^2\left(\frac{\theta_0}{2}\right) = \sqrt{1 - \frac{1}{z_0^{tot.}(x)}}. \tag{5.8}$$

This behavior at x_c produces a first-order transition with the same qualitative characteristics of the four-dimensional model.

5.1 Chemical potentials

A natural and intriguing generalization is to add chemical potentials for the R-charges, while maintaining the trivial vacuum as a gauge background.

The critical equation has still the form (5.6) but $4z_0^{spin.}$ and $6z_0^{scal.}$ are substituted by

$$\begin{aligned} 4z_0^{spin.} &\mapsto \frac{2x}{(1-x)^2} \sum_{p=1}^{4} \left(x^{\frac{1}{4}} y^{-\widetilde{\Omega}_p} + x^{-\frac{1}{4}} y^{\widetilde{\Omega}_p} \right) \\ 6z_0^{scal.} &\mapsto x^{1/2} \frac{x+1}{(1-x)^2} \sum_{p=1}^{3} \left(y^{\Omega_p} + y^{-\Omega_p} \right), \end{aligned} \tag{5.9}$$

which is (4.35) for $q_\alpha = 0$. The effect of small chemical potentials can be easily computed by treating them as a perturbation and expanding around $(\Omega_1, \Omega_2, \Omega_3) = (0, 0, 0)$. This yields the following result

$$\begin{aligned} T_H(\Omega) = T_H(0) &- 0.113946 \sum_{i=1}^{3} \Omega_i^2 - 0.054438 \prod_{i=1}^{3} \Omega_i \\ &- 0.036442 \sum_{i=1}^{3} \Omega_i^4 - 0.014059 \sum_{i<j} \Omega_i^2 \Omega_j^2 + O(\Omega^5), \end{aligned} \tag{5.10}$$

where all the numerical coefficients are actually known exactly, but their explicit expression is long and irrelevant. The presence of small chemical potentials *decreases* the Hagedorn temperature.

In figure 1 we display the dependence of the critical temperature T_H for the three particular choices of critical potential $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, 0, 0)$, $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0)$ and $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, \Omega)$. In all three cases, the behavior around $\Omega = 1$, in a trivial vacuum background, is similar to that of the $\mathcal{N} = 4$ theory in four dimensions discussed in [19, 20].

[8] We are assuming that the relevant features are completely captured by the first mode $n = 1$.

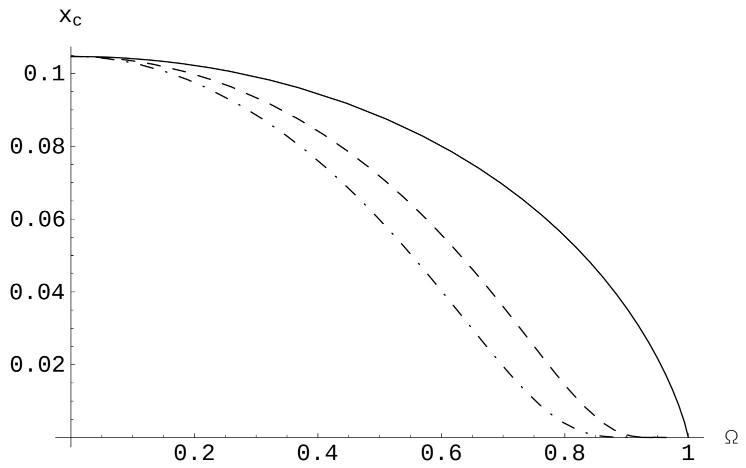

Figure 1: The continuous, dashed and dot-dashed lines correspond to $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, 0, 0)$, $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0)$ and $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, \Omega)$ respectively. All the curves reach $\Omega = 1$ when x_c approaches zero.

We find, in fact:

$$\begin{aligned}
(\Omega_1, \Omega_2, \Omega_3) = (\Omega, 0, 0): \qquad & T_H = -\frac{1}{\log(1-\Omega)}\left[1 - \frac{\log\left(-\log[1-\Omega]\right)}{\log(1-\Omega)} + \dots\right], \\
(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0): \qquad & T_H = \frac{1-\Omega}{\log 2}\left[1 - \frac{1}{\log 2} e^{-\frac{\log 2}{2(1-\Omega)}} + \mathcal{O}(e^{-\frac{\log 2}{(1-\Omega)}})\right], \\
(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, \Omega): \qquad & T_H = \frac{1-\Omega}{\log 4}\left[1 - \frac{30}{\log 4} e^{-2\frac{\log 4}{(1-\Omega)}} + \mathcal{O}(e^{-3\frac{\log 4}{(1-\Omega)}})\right],
\end{aligned} \tag{5.11}$$

which have the same qualitative behavior of the analogous equations found in [20] for the $\mathcal{N} = 4$ theory. This similarity suggests the possibility to consider decoupling limits analogous to those performed in [20] for the $\mathcal{N} = 4$ theory. This might help to single out some subsectors of the present model with simple properties at the (full) quantum level [22–24]. However, this analysis is left for future research.

5.2 High temperatures

In the high temperature regime the eigenvalue distribution becomes almost like a delta-function [9]. Therefore $\rho_n = 1$ and the free energy can be evaluated by looking at the expression of the functional determinants in the background of vanishing flat-connections. When the chemical potentials are strictly zero the leading contribution to the free energy $F = -T \log \mathcal{Z}$ is (see (B.9), (B.14) and (B.15))

$$F = -\frac{7}{\pi}\zeta(3)\, V(S^2) N^2 T^3 + \mathcal{O}(T^2). \tag{5.12}$$

We see that the limiting free energy density here coincides precisely with that of the $\mathcal{N} = 8$ super Yang-Mills theory in flat three-dimensional space. Taking the dimensionless parameter TR to infinity is equivalent to taking the limit of large volume at fixed temperature, loosing in this way any memory of the original deformed supersymmetry. We can also notice that no dependence appears, at the leading order, on the particular monopole vacuum on which the expansion has been performed and the result (5.12) is actually general.

It is interesting to consider the corrections to this result when chemical potentials are taken into account. The first non-trivial contribution is easily evaluated by using the expansions of $\mathrm{Li}_3(z)$ presented in (B.15): we simply notice that chemical potentials appear as imaginary parts of the flat-connections and are contained in the variable z introduced in the appendix B.1. Summing carefully the contributions coming from bosons and fermions, we obtain the free energy

$$F = -V(S^2)N^2T^3\left[\frac{7}{\pi}\,\zeta(3)+\sum_{i=1}^{3}\frac{y_i^2}{4\pi}\left(3-\log\frac{y_i^2}{4}\right)\right]+\mathcal{O}(T^2), \tag{5.13}$$

where we introduced the relevant combination $y_i = \Omega_i/T$. This result is perfectly consistent with the computation performed in [49], for a system of N free D2 branes in the presence of chemical potentials.

6. Thermodynamics in non-trivial vacua I

We shall first consider the multi-matrix model, (4.33), which originates from the *uncharged vacuum*. We recall that in this case the partition function is defined by the matrix integral

$$\mathcal{Z}_A = \int\prod_{I=1}^{k}[dU_I]\exp\left(-\beta V_0+\sum_{IJ}\sum_{n=1}^{\infty}\frac{1}{n}z_{IJ}^{tot.}(x^n)\mathrm{Tr}(U_I^n)\mathrm{Tr}(U_J^{\dagger n})\right), \tag{6.1}$$

where

$$V_0 = r\sum_{\alpha\in\mathrm{roots}}(4|q_\alpha|^2+|q_\alpha|) \tag{6.2}$$

is the Casimir energy. The value of the Casimir energy is puzzling not only for the r dependence, making its sign ambiguous, but also because it depends on the charge of the vacuum q_α so that it is different for different vacua. At the supergravity level we expect instead these vacua to be degenerate. This last feature is reproduced within our second regularization choice, giving a vanishing Casimir energy and consequently degenerate vacua: the price we pay is the introduction of the logarithmic interactions (4.34) that will be studied in the next section.

The large N-limit of the matrix-model (6.1) is investigated by generalizing to a multi-dimensional case the technique presented in the previous section: we introduce the density functions $\rho_I(\theta_I)$ associated to the matrices U_I and in terms of the Fourier-modes ρ_{In}

$$\rho_I(\theta_I) = \frac{1}{2\pi}+\sum_{n=1}^{\infty}(\rho_{In}e^{in\theta_I}+\bar\rho_{In}e^{-in\theta_I}), \tag{6.3}$$

the matrix integral (6.3) reduces as well to an infinite set of independent gaussian integrals

$$\mathcal{Z}_A = \int\prod_{I=1}^{k}D\rho_{In}D\bar\rho_{In}\exp\left(-\beta V_0-N^2\sum_{IJ}\sum_{n=1}^{\infty}\rho_{In}\bar\rho_{Jn}\underbrace{\frac{1}{n}(\delta_{IJ}-z_{IJ}^{tot.}(x^n))s_Is_J}_{V_{IJ}(x^n)}\right), \tag{6.4}$$

where we have introduced the filling fractions $s_I = N_I/N$. In the large N limit (6.4) is dominated by the absolute minimum of the quadratic action

$$S = \sum_{IJ} \sum_{n=1}^{\infty} \rho_{In} \bar{\rho}_{Jn} V_{IJ}(x^n), \tag{6.5}$$

which is given by $\rho_{In} = 0$ for every I and n if the quadratic form $V_{IJ}(x^n)$ is positive definite. For small temperatures, namely small x, the eigenvalues of the matrix $V_{IJ}(x^n)$ are all positive and close to $1/n$ since $V_{IJ}(x^n) \sim \frac{1}{n}\delta_{IJ}$ for $x \ll 1$ (we recall that $z^{tot.}_{IJ}(x^n)$ vanishes as x approaches zero). Therefore the partition function is simply given by the Casimir contribution at the leading order and by the small fluctuation around the minimum at the subleading order

$$\mathcal{Z}_A \propto e^{-\beta V_0} \prod_{n=1}^{\infty} \frac{1}{\det\left(V_{IJ}(x^n)\right)}. \tag{6.6}$$

When we increase the temperature, x approaches 1 and the above description is reliable until the quadratic form $V_{IJ}(x^n)$ develops the first negative eigenvalue. This occurs at the smallest x_c for which one of the eigenvalues of $V_{IJ}(x^n)$ vanishes, or equivalently for which

$$\det\left(V_{IJ}(x_c^n)\right) = 0. \tag{6.7}$$

The smallest $x_c = e^{-1/T_c}$, namely the smallest critical temperature, is obviously obtained for $n = 1$ which provides the strongest condition. Moreover this critical value always exists since $z^{tot.}_{IJ}(x)$ is a monotonic function ranging from 0 to infinity when $x \in [0, 1]$.

We are now ready to investigate the dependence of the critical temperature on the non trivial monopole background. We start by considering a configuration $\mathfrak{f}$ with just two sectors of equal length. It is given by

$$\mathfrak{f} = (n_1, \ldots, n_1, n_2, \ldots, n_2)\ . \tag{6.8}$$

The z_{IJ} and thus the critical temperature depend only on the absolute effective charge, namely $q = |n_1 - n_2|/2$. This property reflects the fact that the global $\mathrm{U}(1)$ sector of charge $(n_1 + n_2)/2$ does not affect the thermodynamics in the large N limit, since there are no degree of freedom which couples to it. We also observe that the critical equation is independent of the filling fractions s_I and it is obtained by requiring the vanishing of the determinant

$$\det\begin{pmatrix} 1 - z^{tot.}_{11}(x) & -z^{tot.}_{12}(x) \\ -z^{tot.}_{21} & 1 - z^{tot.}_{22}(x) \end{pmatrix} = (1 - z^{tot.}_0)^2 - (z^{tot.}_{12})^2 = 0, \tag{6.9}$$

where we have used that the matrix V_{IJ} is symmetric ($z^{tot.}_{12} = z^{tot.}_{21}$) and that $z^{tot.}_{11} = z^{tot.}_{22} = z^{tot.}_0$ is the partition function in the trivial vacuum. This equation naturally splits into two simpler equations

$$(a):\qquad \lambda_-(x) = 1 - z^{tot.}_0(x) - z^{tot.}_{12}(x) = 0 \tag{6.10}$$

$$(b):\qquad \lambda_+(x) = 1 - z^{tot.}_0(x) + z^{tot.}_{12}(x) = 0. \tag{6.11}$$

q	x_c	T_c
1/2	0.085786	0.407183
1	0.099771	0.433863
3/2	0.103842	0.441523
2	0.104567	0.442884
5/2	0.104672	0.443081
3	0.104686	0.443107
7/2	0.104688	0.443111

Table 1: x_c and T_c in the two sectors situation as a function of the relative monopole charge q.

The critical temperature is determined by the lowest zero of these two equations. Since $\lambda_+ - \lambda_- = 2z_{12} \geq 0$ and $\lambda_+(0) = \lambda_-(0) = 1$, $\lambda_-(x)$ reaches its zero at a smaller temperature: in determining x_c we can then neglect $\lambda_+(x)$.

From the structure of the critical equation, $\lambda_-(x) = 0$, we can deduce two general properties of the critical temperature. First, the positivity of $z_{12}^{\rm tot}$ also ensures that $\lambda_-(x) \leq \lambda_0(x) = (1 - z_0^{\rm tot})$. This means that the critical temperature in a non-trivial monopole background will always be smaller than the corresponding one in the trivial vacuum. Second, the function $z_{12}^{tot.}$ decreases with the monopole charge q (in the interval $x \in [0,1]$): this implies that the critical temperature increases with the monopole charge. When q approaches infinity the value of the critical temperature becomes that of the trivial vacuum. Below we present a table for the critical temperature, where the behaviors described above are manifest

When the number k of sectors grows, the dependence of the critical temperature T_c on the relative monopole charges becomes quite intricate. However, some general behaviors can be anticipated. Consider, for example, a generic background of the form

$$\mathfrak{f} = (n_1, \ldots, n_1, n_2, \ldots, n_2, \ldots\ldots, n_k, \ldots, n_k), \tag{6.12}$$

where the induced relative monopole charges

$$q_{IJ} = \frac{|n_I - n_J|}{2} \tag{6.13}$$

are large, namely n_I and n_J are very different from each other. Then the Hagedorn temperature is dominated by the smallest charge and the off-diagonal terms associated to the other charges can be considered as small perturbations. The determinant is approximately given by

$$\det(V_{IJ}) \approx (1 - z_0)^{k-2}((1 - z_0^{\rm tot})^2 - (z_{q_{\rm min}}^{\rm tot})^2). \tag{6.14}$$

Exploiting what we have learned for the $k = 2$ system, the lowest transition temperature is an approximate solution of the equation $1 - z_0^{\rm tot} - z_{q_{\rm min}}^{\rm tot} = 0$.

Another interesting family of configurations is built by considering long sequences of sectors with equal length and monopole charge increasing by a fixed value $\mathfrak{q}$, namely

$$\mathfrak{f} = (n_0, \ldots, n_0, n_0 + \mathfrak{q}, \ldots, n_0 + \mathfrak{q}, n_0 + 2\mathfrak{q}, \ldots, n_0 + 2\mathfrak{q}, \ldots\ldots, n_0 + k\mathfrak{q}, \ldots, n_0 + k\mathfrak{q}). \tag{6.15}$$

k	x_c	T_c
2	0.085786	0.407184
3	0.079653	0.395245
10	0.072873	0.381820
15	0.072312	0.380697
20	0.072098	0.380267
30	0.071936	0.379942
60	0.071833	0.379736

Table 2: x_c and T_c in the k sectors situation at $\Omega = 0$. The vacua are labelled by $\mathfrak{f}_k = \mathrm{diag}(k-1, \ldots, k-2, \ldots, 0)$.

When the number of sectors k goes to infinity, the Hagedorn temperature in these vacua approaches that of $\mathcal{N} = 4$ on the Lens space $S^3/\mathbb{Z}_\mathfrak{q}$ in the sector described by a vanishing flat-connection. For example for $\mathfrak{q} = 1$, a simple numerical analysis shows that T_c goes to that of pure $\mathcal{N} = 4$, $T_c^{D=4} = -1/\log(7-4\sqrt{3}) \simeq 0.379663$ [9], (see table below). Analytically, this result can be argued by noting that the matrix V_{IJ}, of which we have to compute the determinant, is of Toeplitz type, namely a matrix in which each descending diagonal from left to right is constant. Consequently its entries do not depend on I and J separately, but only on the difference $I - J$. For this kind of matrices, when the dimension is large, the determinant is approximated by that of their circulant version [50]. This means that the smallest zero of the determinant can be found as a solution of

$$1 - \sum_{k=-\infty}^{\infty} z^{tot.}(k\mathfrak{q}, x) = 0, \tag{6.16}$$

which is the smallest eigenvalue of the corresponding circulant matrix. In (6.16) $z^{tot.}(k\mathfrak{q}, x)$ is the single-particle partition function in the sector of charge $k\mathfrak{q}$. It is now possible to show that this infinite sum produces the single-particle partition function of the $\mathcal{N} = 4$ SYM theory in the trivial vacuum of $S^3/\mathbb{Z}_\mathfrak{q}$ (see [39] for comparison). In other words (6.16) coincides with the critical equation for the $\mathcal{N} = 4$ SYM theory in the trivial vacuum of $S^3/\mathbb{Z}_\mathfrak{q}$.

Finally we consider the addition of chemical potentials to a monopole configurations. Their introduction does not alter significantly the picture and a numerical analysis is given in figure 2.

6.1 Just above the critical temperature

To understand what happens when we cross the critical temperature, we shall now focus our attention on the two-sectors configuration (6.8). In this case, if we introduce the combination

$$\rho_\pm = \frac{1}{2}(\rho_1 \pm \rho_2)\,, \tag{6.17}$$

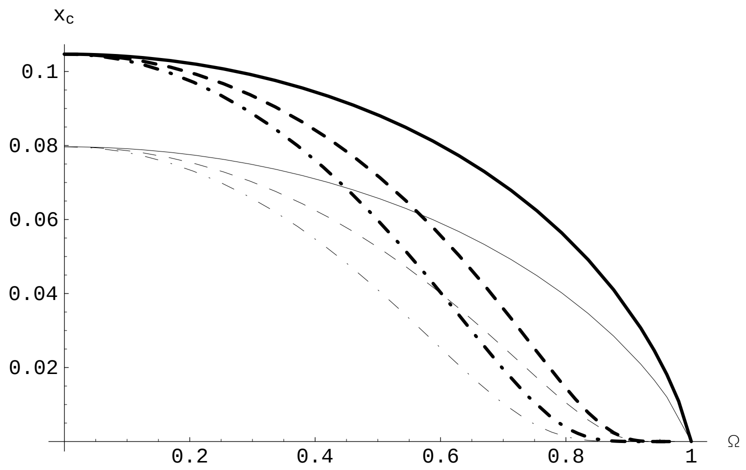

Figure 2: Transition lines for three sectors vacuum: $\mathfrak{f} = (i, \frac{N}{3}; 0, \frac{N}{3}; -i, \frac{N}{3})$. Narrow lines corresponds to $i = 1$, thick lines to $i = 10$. The convention for continuous, dashed and dot-dashed are those of figure 1. The qualitative behavior is the same for every number of sectors.

the action takes a diagonal form

$$S = 2\sum_{n=1}^{\infty}\left(\frac{1}{n}\lambda_-(x^n)\bar{\rho}_{+n}\rho_{+n} + \frac{1}{n}\lambda_+(x^n)\bar{\rho}_{-n}\rho_{-n}\right). \tag{6.18}$$

Above the critical temperature, $\lambda_-(x)$ is negative and the dominant saddle-point is no longer realized by a flat distribution $\rho_{1n} = \rho_{2n} = 0$ ($\rho_{+n} = \rho_{-n} = 0$). In fact, as the temperature is increased, the attractive term in the pairwise potential continues to increase in strength, so the eigenvalues become increasingly bunched together, occupying, at the end, only a finite interval $I = [-\theta_0, \theta_0]$ on the circle (we arbitrarily choose the middle of this interval to be at $\theta = 0$ for convenience). However, since $\lambda_+(x)$ is still positive, we can safely assume that the new dominant saddle point satisfies

$$\rho_{-n} = \frac{1}{2}(\rho_{1n} - \rho_{2n}) = 0, \qquad \text{i.e. } \rho_1 = \rho_2 = \rho_+ \,. \tag{6.19}$$

In other words, the problem reduces to an effective one matrix model governed by the action

$$\begin{aligned} S =& 2\int d\theta d\theta' \rho_+(\theta)\rho_+(\theta')\sum_{n=1}^{\infty}\left[\frac{(\lambda_-(x^n) - 1)}{n}\cos(n(\theta-\theta'))\right] + \\ &+ 2\int d\theta d\theta' \rho_+(\theta)\rho_+(\theta')\log\left|\sin\frac{\theta-\theta'}{2}\right|, \end{aligned} \tag{6.20}$$

where the distribution function has support in the interval $[-\theta_0, \theta_0]$. In complete analogy with what we found in trivial vacuum case (6.21), we have

$$\rho(\theta) = \frac{\cos\left(\frac{\theta}{2}\right)}{\pi\sin^2\left(\frac{\theta_0}{2}\right)}\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} \quad \text{with} \quad \cos^2\left(\frac{\theta_0}{2}\right) = \sqrt{\frac{\lambda_-(x)}{\lambda_-(x)-1}}. \tag{6.21}$$

Near the critical temperature, for $T > T_H$ we have the following expansion for the partition function

$$\frac{F}{N^2} = \frac{T_H}{2}\lambda_-(x) + \mathcal{O}((\lambda_-)^2) = \frac{T_H}{2}(T - T_H)\frac{\partial\lambda_-}{\partial T}\bigg|_{T=T_H} + \mathcal{O}((T - T_H)^2), \tag{6.22}$$

which gives the characteristic first-order transition, already found in the four-dimensional model.

7. Thermodynamics in non-trivial vacua II

We discuss now the thermodynamical behavior arising when the second regularization scheme, considered for the fermions in section 4, is adopted. As previously derived, a non-trivial logarithmic deformation of the multi-matrix model (6.1) has to be considered

$$\mathcal{Z}_B = \int \prod_{I=1}^{k}[dU_I] \exp\left(\sum_{IJ}\sum_{n=1}^{\infty}\frac{1}{n} z_{IJ}^{tot.}(x^n)\mathrm{Tr}(U_I^n)\mathrm{Tr}(U_J^{\dagger n})\right)\prod_{I=1}^{k}\det(U_I)^{(Nn_I - Q)}. \tag{7.1}$$

To illustrate the effect of the new interactions on the large N dynamics, we shall make a very drastic assumption and we shall focus our attention just on the first winding, $n = 1$. With this choice the original matrix integral reduces to

$$\int \prod_{I=1}^{k} DU_I \exp\left(\sum_{IJ} z_{IJ}^{tot.}(x)\mathrm{Tr}(U_I)\mathrm{Tr}(U_J^{\dagger})\right)\prod_{I=1}^{k}\det(U_I)^{Nn_I - Q}. \tag{7.2}$$

It is useful, as a first step, to introduce a set of k complex Lagrange multipliers λ_I and the partition function can be written as

$$\begin{aligned}&\frac{\prod_{I=1}^{k} N_I^2}{(\det(z_{IJ}))^k}\int\prod_{J=1}^{k} d\lambda_J d\bar{\lambda}_J \exp\left(-\sum_{IJ} N_I N_J \bar{\lambda}_I z_{IJ}^{-1}(x)\lambda_J\right)\times\\ &\prod_{I=1}^{k}\int DU_I \exp\left(\bar{\lambda}_I N_I \mathrm{Tr}(U_I) + \lambda_I N_I \mathrm{Tr}(U_I^{\dagger})\right)\det(U_I)^{Nn_I - Q}.\end{aligned} \tag{7.3}$$

Next we use the polar decomposition $\lambda_I = \gamma_I e^{i\alpha_I}$ for each Lagrange multipliers. The phases $e^{i\alpha_I}$ are then decoupled from the matrix integration by means of the change of variables $U_I \mapsto U_I e^{i\alpha_I}$. This procedure yields the following integral

$$\begin{aligned}&\frac{\prod_{I=1}^{k} N_I^2}{(\det(z_{IJ}))^k}\int\prod_{J=1}^{k} d\gamma_I d\alpha_I \exp\left(-\sum_{IJ} N_I N_J \gamma_I z_{IJ}^{-1}(x)e^{i(\alpha_J - \alpha_I)}\gamma_J + i\sum_{I=1}^{k} N_I(Nn_I - Q)\alpha_I\right)\\ &\qquad\times\prod_{I=1}^{k}\int DU_I \exp\left(\gamma_I N_I(\mathrm{Tr}(U_I) + \mathrm{Tr}(U_I^{\dagger}))\right)\det(U_I)^{Nn_I - Q}.\end{aligned} \tag{7.4}$$

In (7.4) the group integrations over the unitary matrices U_I are completely decoupled. Each matrix integration corresponds to a Gross-Witten model [35] with a coupling γ_I and an additional logarithmic potential proportional to $\log(\det(U_I))$. We remark that these kinds of deformations for unitary matrix models were widely considered in the early eighties (see e.g. [27, 28]). The determinant operator was expected to act as an order parameter for the large N phase transitions characterizing this class of models [51]: unfortunately, we cannot simply borrow the old results. In (7.4) in fact we have a new and decisive ingredient with respect to the original investigations: the power of the determinant is not a fixed number, but it grows linearly with N. This last feature dramatically alters the usual large N dynamics since the integral (7.4) is not dominated anymore by the same family of saddle-points of the familiar Gross-Witten model, as we will see in the following.

7.1 Solution of unitary matrix model with logarithmic potential

The phase structure of (7.4) can be naturally studied along the lines proposed in [52]. We will first perform the integration over the unitary matrices and then the integration over the Lagrange multipliers. We will then start by studying the large N properties of the reduced model

$$\mathcal{Z}(\gamma,p)=\int DU \exp\Big(\gamma N(\mathrm{Tr}(U)+\mathrm{Tr}(U^\dagger))\Big)\det(U)^{Np}, \tag{7.5}$$

where Np is an integer, whose sign is irrelevant because we can transform Np into $-Np$ by performing the change of variable $U\mapsto U^\dagger$. For this reason, from now on, we shall take p to be positive. The first important effect of the new logarithmic interaction concerns the small γ behavior of (7.5): differently from the Gross-Witten model ($p=0$), where $\mathcal{Z}(\gamma,p)$ is finite as γ approaches zero, here $\mathcal{Z}(\gamma,p)$ vanishes as γ^{N^2p}. This leading behavior is determined by expanding the exponential around $\gamma=0$ and performing the integral term by term. The first non-vanishing contribution is fixed by the selection rule imposed by the U(1) factor present in U(N) and it is given by

$$\mathcal{Z}(\gamma,p)\approx\frac{(\gamma N)^{N^2p}}{(N^2p)!}\int DU\ \mathrm{Tr}(U^\dagger)^{N^2p}\det(U)^{Np}=(\gamma N)^{N^2p}\prod_{i=0}^{N-1}\frac{i!}{(i+Np)!}=(2\gamma)^{N^2p}e^{N^2C}, \tag{7.6}$$

where the constant C in the large N limit is given by

$$C=-\frac{1}{2}\left((\log(4)-3)p+(p+1)^2\log(p+1)-p^2\log(p)\right). \tag{7.7}$$

In other words, the free energy $\mathcal{F}(\gamma,p)=\log\mathcal{Z}(\gamma,p)=N^2\mathcal{F}_0(\gamma,p)+\ldots$ of the present unitary matrix model starts, at leading N^2 order, with a logarithmic singularity similar to the one of the usual Penner model [53, 54]

$$\mathcal{F}_0(\gamma,p)=p\log(2\gamma)+C+O(\gamma^2). \tag{7.8}$$

This new behavior suggests that the usual strong-coupling expansion of (7.5) might be radically different from that of the Gross-Witten model, which is simply given by $e^{N^2\gamma^2}$. To explore this idea, one could perform a full strong-coupling expansion and to resum the resulting series in the large N limit; however the presence of the determinant factor much complicates this approach. Here, we shall choose a simpler path and consider a different expansion, peculiar of the present model, namely p very large. In this limit we can perform a semiclassical analysis on the integral (7.5): the relevant classical potential is, in this case,

$$pV(\theta_i) = pN\left(2\frac{\gamma}{p}\sum_{i=1}^{N}\cos\theta_i + i\sum_{i=1}^{N}\theta_i\right). \tag{7.9}$$

The equations for the critical point are easily derived and solved (we will denote from now on $4\gamma^2 = t$)

$$-\frac{\sqrt{t}}{p}\sin\theta_i + i = 0 \quad \Rightarrow \quad \theta_i = i\sinh^{-1}\left(\frac{p}{\sqrt{t}}\right). \tag{7.10}$$

The semiclassical approximation is then obtained by expanding the classical action around the critical point up to the quadratic order

$$N^2\left(\sqrt{p^2+t} - p\sinh^{-1}\left(\frac{p}{\sqrt{t}}\right)\right) - \frac{N}{2}\left(\sqrt{p^2+t}\right)\sum_{i=1}^{N}\hat{\theta}_i^2 + O\left(\hat{\theta}_i^3\right), \tag{7.11}$$

with

$$\hat{\theta}_i \equiv \theta_i - i\sinh^{-1}\left(\frac{p}{\sqrt{t}}\right). \tag{7.12}$$

We remark that this is a good approximation as long as $\sqrt{p^2+t} \gg 1$: in this limit the gaussian integration covers the whole real line and the Haar measure over the unitary matrices becomes the usual measure over the hermitian matrices. We can easily perform the integration over the angles θ_i and up to a constant independent of p we get

$$\begin{aligned}\mathcal{F}_0(t,p) &= \left(\sqrt{p^2+t} - p\sinh^{-1}\left(\frac{p}{\sqrt{t}}\right) - 1/2\log(\sqrt{p^2+t})\right) = \\ &= \left(\sqrt{p^2+t} - p\log\left(\frac{p}{\sqrt{t}} + \sqrt{\frac{p^2}{t}+1}\right) - 1/2\log(\sqrt{p^2+t})\right).\end{aligned} \tag{7.13}$$

For p large and t finite or small, we finally arrive to the following expansion

$$\begin{aligned}\mathcal{F}_0(t,p) =& p\left(\frac{\log(t)}{2} - \log(p) - \log(2) + 1\right) - \frac{1}{2}\log(p) + \frac{t}{4p} - \frac{1}{4}t\left(\frac{1}{p}\right)^2 - \\ &- \frac{1}{32}t^2\left(\frac{1}{p}\right)^3 + \frac{1}{8}t^2\left(\frac{1}{p}\right)^4 + O\left(\left(\frac{1}{p}\right)^5\right).\end{aligned} \tag{7.14}$$

This result is quite remarkable: we see that the above expansion reproduces exactly the large p limit of (7.8) and contains a infinite series of corrections in powers of t. Since (7.14) holds also for small t, we must conclude that the strong-coupling expansion of our deformed Gross-Witten model leads to a non-trivial function of t and p, eventually encoding an intriguing modification of the $p = 0$ result.

It is quite easy to repeat the same analysis taking t large, exploring in this way the deformation of the weak-coupling phase of the familiar unitary model. In this limit we should obtain, at leading order in t, the very same result for the free energy as in the Gross-Witten case: again we could expect a non-trivial deformation due to the presence of p. Actually, performing the same steps as before, we get again (7.13),[9] which expanded for large t gives

$$\begin{aligned}\mathcal{F}_0(t,p) = &-\frac{3}{4}+\sqrt{t}-\frac{1}{4}\log(t)-\frac{1}{2}p^2\sqrt{\frac{1}{t}}-\frac{p^2}{4t}+\frac{1}{24}p^4\left(\frac{1}{t}\right)^{3/2}+\frac{1}{8}p^4\left(\frac{1}{t}\right)^2-\\ &-\frac{1}{80}p^6\left(\frac{1}{t}\right)^{5/2}-\frac{1}{12}p^6\left(\frac{1}{t}\right)^3+O\left(\left(\frac{1}{t}\right)^{7/2}\right).\end{aligned} \tag{7.15}$$

We recognize in the first three terms the exact large N result of the Gross-Witten weak-coupling phase: it does not come as a surprise, being the semiclassical approximation exact in this phase. As expected, we also observe an infinite series of corrections, depending on p, that modify non-trivially the usual spherical free energy of the weak-coupling phase.

We do not expect, of course, that the above expansions yield the exact large N free energy: these results are semiclasssical, in the sense that we missed the contribution of the Vandermonde determinants associated to the measure over *unitary* matrices, that is essential in recovering the correct spherical free energy. Nevertheless they should capture the leading order behavior at large p or t of the complete large N answer, and also a certain series of subleading terms (as we will explicitly check in the following).

These computations suggest an intriguing possibility: we observe non-trivial deformations of both strong and weak-coupling expansion of the Gross-Witten model, involving complicated functions of p and t. It is quite natural to conjecture, at this point, that a unique non-trivial analytic function $\mathcal{F}_0(t,p)$ exists, reproducing for $p \neq 0$ both behaviors and being the large N free energy of the model. This is also suggested by the fact that the same free energy (7.13) describes smoothly either the large p or the large t region (see footnote 9). If this is the case, the presence of the logarithmic interaction would smooth out the third-order phase transition of the Gross-Witten model, the parameter p providing an analytic interpolation between the strong and the weak-coupling phase.

In order to prove this idea, we have to solve exactly the large N dynamics: we shall exploit the beautiful relation between our model and the Painlevé III system illustrated in [29]. In that paper the authors have shown that it is possible to construct an auxiliary function,

$$\sigma(t) = -t\frac{d}{dt}\log\left((tN^2)^{N^2p^2/2}e^{-N^2t/4}\mathcal{Z}(t,p)\right), \tag{7.16}$$

that satisfies, at finite N, the following non-linear differential equation

$$-\frac{1}{16}p^2N^6+\left(p^2-1\right)\sigma'(t)^2N^2+\sigma'(t)\left(4\sigma'(t)-N^2\right)\left(\sigma(t)-t\sigma'(t)\right)+t^2\sigma''(t)^2=0. \tag{7.17}$$

In the large N limit, the spherical ansatz for the partition function $\mathcal{Z}(t,p) = e^{N^2\mathcal{F}_0(t,p)}$ dictates the following scaling for the auxiliary $\sigma(t)$

$$\sigma(t) = N^2\rho(t). \tag{7.18}$$

[9] The semiclassical computation really holds for $\sqrt{p^2+t} \gg 1$ and this condition is realized by taking either p or t large. Therefore we have to obtain the same free energy (7.13) in the large t case as well.

Thus, at the leading order in N^2, we obtain a nice first-order differential equation for the reduced function $\rho(t)$

$$-4t\rho'(t)^3+\left(p^2+t+4\rho(t)-1\right)\rho'(t)^2-\rho(t)\rho'(t)-\frac{p^2}{16}=0. \tag{7.19}$$

The analysis for small and large t given in (7.8) and (7.15) provides two possible boundary conditions for the above equation:

(s): $\rho(t)|_{t=0}=-\frac{1}{2}(p^2+p)$;

(w): $\rho(t)|_{t\to\infty}=\frac{t}{4}-\frac{1}{2}\sqrt{t}$.

Since (7.19) is a first-order differential equation, these boundary values will correspond, in general, to two different solutions: the former, which satisfies (s), is denoted with $\rho_s(t)$ and it is supposed to describe the strong-coupling regime;[10] the latter, $\rho_w(t)$, obeys (w) and it is expected to hold in the weak-coupling regime. The two corresponding free energies $\mathcal{F}_0^{s,w}(t,p)$ are then constructed by integrating the simple relation

$$\frac{d\mathcal{F}_0^{s,w}(t,p)}{dt}=\left(\frac{1}{4}-\frac{p^2}{2t}-\frac{\rho_{s,w}(t)}{t}\right) \tag{7.20}$$

which follows from (7.16) once we have used the spherical ansatz $\mathcal{Z}(t,p)=e^{N^2\mathcal{F}_0^{s,w}(t,p)}$.

The above simple picture works very well at $p=0$, where our model reduces to the usual Gross-Witten model. In this case the differential equation becomes extremely tractable, factorizing into two simple first-order equations: the solution $\mathcal{F}_0^s(t,0)$ and $\mathcal{F}_0^w(t,0)$ can be obtained explicitly and they exactly coincides with the well-known free energies of the model at strong and weak coupling. The condition $\mathcal{F}_0^s(t,0)=\mathcal{F}_0^w(t,0)$ defines the correct critical value for the coupling constant ($t_c=1$). When $p\neq 0$, the situation reserves some surprises as we shall illustrate below.

As thoroughly described in appendix D, the general case can be solved exactly, in spite of the apparent difficult non-linearity of the differential equation. In particular there are two relevant solutions, describing respectively the deformations of $\rho_s(t)$ and $\rho_w(t)$ found in the Gross-Witten case. Integrating (7.20) we get a candidate $\mathcal{F}_0^s(t,p)$ given by

$$\mathcal{F}_0^s(t,p)=-\frac{1}{2}\left((\log(4)-3)p+(p+1)^2\log(p+1)-p^2\log(p)\right)+\frac{t}{4(1+p)}-\frac{p}{2}\log(t)\ , \tag{7.21}$$

while $\mathcal{F}_0^w(t,p)$ has the form

$$\begin{aligned}\mathcal{F}_0^w(t,p)=&f_w+\left(\frac{p^2}{4\rho'_w}-\frac{p^2}{64\left(\rho'_w\right)^2}+\frac{1}{2}\left(\log\left(\rho'_w\right)p^2-2p\tanh^{-1}\left(p+4\left(\frac{1}{p}-p\right)\rho'_w\right)+\right.\right.\\&\left.\left.+\log\left(1-4\rho'_w\right)+\frac{2}{1-4\rho'_w}\right)\right),\end{aligned} \tag{7.22}$$

[10] In the matrix model language, γ is conventionally identified with the inverse of the fundamental coupling constant. Thus small values of $t=4\gamma^2$ are in the strong-coupling region.

where the constant f_w is given by

$$f_w = -\frac{3}{4} + \frac{1}{4}p((-3+\log(16))p - 2\log(p-1) + 2\log(p+1)). \tag{7.23}$$

Here $\rho'_w(t)$ is the solution of the fourth order algebraic equation (D.4), which respects the large t behavior implied by the boundary condition (w). One can easily check that $\mathcal{F}_0^w(t,p)$ smoothly reduces, as p goes to zero, to the free energy of the Gross-Witten model in the weak-coupling phase, and accurately reproduces the semiclassical expansion (7.15), up to higher order terms in $p^{2n}/t^{n+m/2}$, coming from the exact large N solution encoded into the differential equation. It is also evident from (7.21) that $\mathcal{F}_0^s(t,p)$ reproduces, in the limit of vanishing p, the Gross-Witten strong-coupling result.

On the other hand, we already know that $\mathcal{F}_0^s(t,p)$, as given by (7.21), *cannot* provide the right solution describing the small t regime! The large p expansion of (7.21) is quite boring and does not reproduce the non-trivial series (7.14), obtained from the semiclassical approximation. On the other hand it is possible to show that $\mathcal{F}_0^w(t,p)$, for $p \neq 0$, also satisfies the right boundary condition to describe the strong-coupling region (see appendix D) and, more importantly, correctly reproduces (7.14) in the large p limit (up to higher order corrections in t^n/p^{2n+m}, coming from the exact large N solution of the model).

We arrive therefore to the conclusion that the critical behavior of the standard unitary matrix model is completely modified by the addition of our logarithmic interaction. As long as $p \neq 0$ the system is always in a "weak-coupling" phase, described by the free energy $\mathcal{F}_0^w(t,p)$: this solution has the correct boundary condition both at small and at large t and smoothly interpolates between them. We also identify $\mathcal{F}_0^s(t,p)$ with an *unphysical* solution of the differential equation (7.19) and therefore we neglect it. The situation drastically changes for $p=0$: it is possible to show that, starting from $\mathcal{F}_0^w(t,p)$, the limiting behavior changes discontinuously at $t=1$. On the other hand, taking $p=0$ at level of the differential equation (7.19), the strong-coupling phase is instead encoded into the solution $\rho_s(t)$.

7.2 Phase-structure in non-trivial vacua

In this subsection, we shall explore the consequences of the previous results on the phase structure of the theory. After having performed the integration over the unitary matrices in the deformed Gross-Witten models, we are left with the integration over the Lagrange multipliers

$$\int \prod_{J=1}^{k} d\gamma_I d\alpha_I \exp\left(N^2 \sum_{I=1}^{k}(is_I(n_I-q)\alpha_I + s_I^2\mathcal{F}_0(\gamma_I,p)) - N^2\sum_{IJ} s_I s_J \gamma_I z_{IJ}^{-1}(x) e^{i(\alpha_J-\alpha_I)}\gamma_J\right) \tag{7.24}$$

with $q = Q/N$. Since N is large, we can perform this integral in the saddle-point approximation as well. The saddle-points which dominate this integration are determined by

$$
\begin{aligned}
&2\sum_{I=1}^{k} s_I\gamma_I z^{-1}_{IJ}\gamma_J\sin(\alpha_J-\alpha_I)+i(n_J-q)=0\\
&-2\sum_{I=1}^{k} s_I\gamma_I z^{-1}_{IJ}\cos(\alpha_J-\alpha_I)+s_J\mathcal{F}_0'(\gamma_J,p)=0.
\end{aligned}
\tag{7.25}
$$

To be concrete, we shall consider only the case $k=2$: here the relevant combinations of the parameters are given by $n_1-q=s_2(n_1-n_2)\equiv s_2 n$ and $n_2-q=-s_1(n_1-n_2)\equiv -s_1 n$ ($n=n_1-n_2>0$). The first equation in (7.25) then produces two conditions

$$
2\gamma_2 z^{-1}_{21}\gamma_1\sin(\alpha_1-\alpha_2)+in=0 \quad\text{and}\quad 2\gamma_1 z^{-1}_{12}\gamma_2\sin(\alpha_2-\alpha_1)-in=0. \tag{7.26}
$$

These two equations are obviously equivalent and they are solved by

$$
\sin(\alpha_1-\alpha_2)=-i\frac{n}{2\gamma_2 z^{-1}_{21}\gamma_1}\qquad\Rightarrow\qquad \cos(\alpha_1-\alpha_2)=\pm\sqrt{1+\frac{n^2}{4(\gamma_2 z^{-1}_{21}\gamma_1)^2}}. \tag{7.27}
$$

Substituting this result into (7.25), the second equation provides two relations, which determines γ_1,γ_2

$$
\begin{aligned}
&-2s_1\gamma_1^2 z^{-1}_{11}\mp 2s_2\sqrt{(\gamma_2 z^{-1}_{21}\gamma_1)^2+\frac{n^2}{4}}+s_1\gamma_1\mathcal{F}_0'(\gamma_1,p)=0,\\
&\mp 2s_1\sqrt{(\gamma_2 z^{-1}_{21}\gamma_1)^2+\frac{n^2}{4}}-2s_2\gamma_2^2 z^{-1}_{11}+s_2\gamma_2\mathcal{F}_0'(\gamma_2,p)=0.
\end{aligned}
\tag{7.28}
$$

In the following, we shall further simplify our example and we shall choose two sectors of equal length, namely we shall set $s_1=s_2=1/2$. Then, by taking the difference of the two equations and using the fact that $\mathcal{F}_0(t,p)$ is a monotonic function, one can immediately show that $\gamma_1=\gamma_2$. We remain with just one equation, which determines $t_1=4\gamma_1^2$

$$
\mp\sqrt{\frac{1}{4}(z^{-1}_{12}t_1)^2+n^2}-\frac{t_1}{2}z^{-1}_{11}+2t_1\mathcal{F}_0'(t_1,p)=0, \tag{7.29}
$$

which is conveniently rewritten in terms of $\rho_w(t_1)$ as follows

$$
f_\pm(t_1)\equiv\pm\sqrt{\frac{1}{4}(z^{-1}_{12}t_1)^2+n^2}+\frac{t_1}{2}(1-z^{-1}_{11})+n=2\left(\rho_w(t_1)+\frac{n}{2}(n+1)\right). \tag{7.30}
$$

When t_1 runs from zero to infinity, the r.h.s of (7.30) spans the same region. Thus a necessary condition for having a non-trivial solution is that the l.h.s. of (7.30) is not negative definite. Let us discuss the first equation: $f_+(t)$ has the following properties

$$
\begin{aligned}
&f_+(0)=2n &&f_+'(0)=\frac{1}{2}\frac{\det z-z_{11}}{\det z}\\
&f_+'(t)=0\quad\Rightarrow &&t^2=-4n^2\frac{(\det z-z_{11})^2}{(z_{12})^2}\frac{\det z}{\det(1-z)}.
\end{aligned}
\tag{7.31}
$$

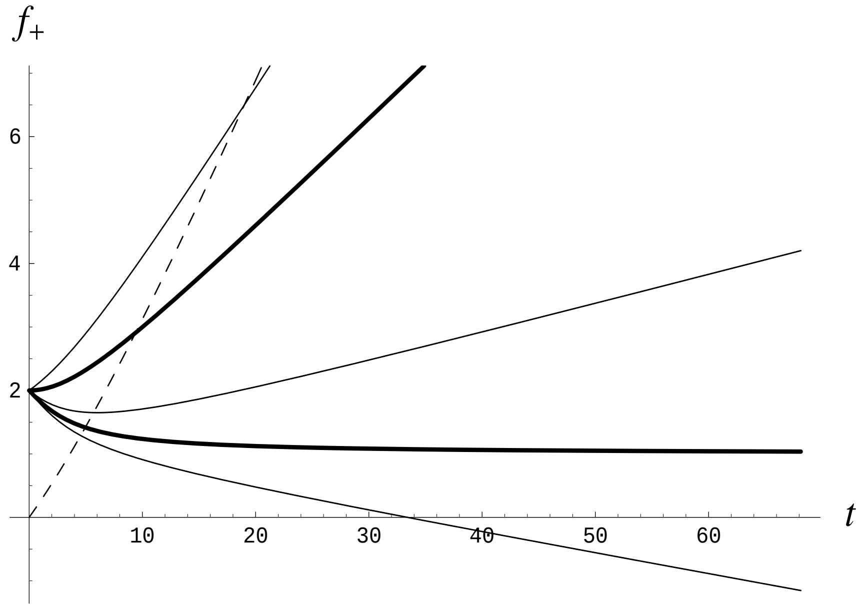

Figure 3: Plot of $f_+(t)$ for different values of T and $n = 1$, $q = 1/2$. Going bottom-up, the solid lines illustrate the behavior for $T < T_H$, $T = T_H$ (lower thick line), $T_H < T < T_2$, $T = T_2$ (upper thick line), $T > T_2$. The dashed line is the r.h.s. of (7.30) as a function of t.

Moreover we have that for large t

$$f_+(t) \to \frac{t}{2}\frac{z_{11} + z_{12} - 1}{z_{11} + z_{12}} + n + \mathcal{O}(1/t). \tag{7.32}$$

We immediately conclude that for temperatures near zero ($x \ll 1$), $f_+(t)$ is always decreasing: for $T < T_H$, where T_H is the Hagedorn temperature defined by the equation

$$z_{11} + z_{12} = 1 \tag{7.33}$$

as in (6.10), we have that $\det(1 - z) \geq 0$, implying that $f'_+(t)$ never vanishes for $t > 0$. Therefore in this range of temperature there is always one solution to the saddle-point equation at $t \neq 0$. At the Hagedorn temperature T_H the function $f_+(t)$ is still decreasing but becomes positive definite, asymptotically approaching the value n. Above T_H we see that $f_+(t)$ develops a minimum at finite t and then becomes monotonically increasing. The minimum disappears at the temperature T_2 defined by

$$\det z = z_{11} \tag{7.34}$$

and the function becomes monotonically increasing for any $T > T_2$.

In spite of these changes of behavior, one can check that there is always one solution to the saddle-point equation as shown, in different regimes, in figure 3. Moreover the position of this saddle-point changes smoothly as function of the temperature (figure 4).

Let us examine the second saddle-point equation, the one involving $f_-(t)$. We can repeat the same analysis: the main conclusion is that for temperature $T < T_2$ we see $f_-(t)$

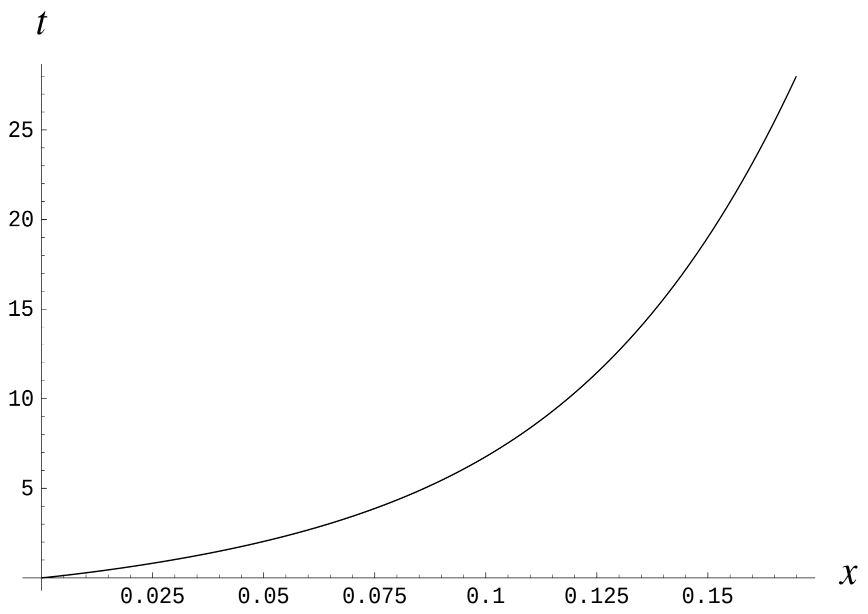

Figure 4: Saddle point as a smooth function of the temperature x for $n = 1$, $q = 1/2$. The graph covers all the different regimes, the Hagedorn temperature being $x_H = 0.0857864$ and T_2 corresponding to $x_2 = 0.115493$.

being monotonically decreasing and therefore, because $f_-(0) = 0$, there is no solution for $t \neq 0$ to the saddle-point equation. We notice that $t = 0$ is not acceptable because of (7.26). For $T > T_2$ it is not easy to see analytically if $f_-(t)$ provides new solutions to the saddle-point equation: we have done a numerical study, showing that a new solution appears for $x \geq 0.212352$. However, the resulting free energy is always subdominant with respect to the other one as illustrated in figure 5. So the solution associated to f_+ is the only relevant saddle-point in the large N limit.

We conclude therefore that within our approximation, that consisted in taking just the first winding in the matrix model action ($n = 1$), we have always a non-trivial saddle-point giving a free energy $F_B = \log \mathcal{Z}_B$ of order N^2. Moreover this saddle-point varies continuously with the temperature: in particular at the Hagedorn temperature T_H, representing the point of the first-order phase transition in our first regularization scheme, the free energy remains smooth and no discontinuous behavior appears in this second scheme.

8. Conclusions and future directions

In this paper we have studied the maximal supersymmetric gauge theory on $\mathbb{R} \times S^2$, with particular attention to its thermodynamical properties in the limit of zero 't Hooft coupling. In the case of the trivial vacuum, we found a behavior similar to the parent four-dimensional theory, with a first-order Hagedorn transition separating a "confining" phase from a "deconfined" one, with non-trivial expectation value for the Polyakov loop. We have repeated the analysis for monopole vacua and we have apparently different behaviors,

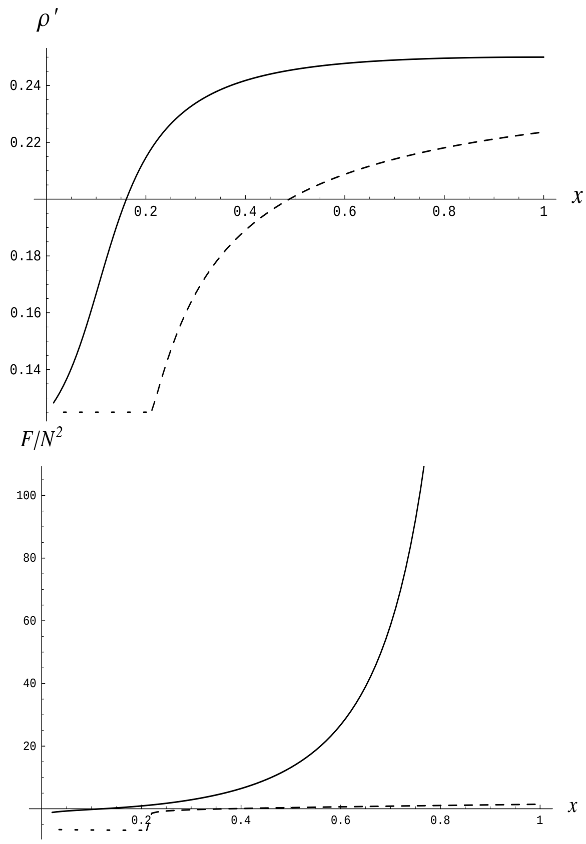

Figure 5: In the upper graph the saddle-points (for $n = 1$, $q = 1/2$) in terms of $\rho'(x)$ associated to f_+ (continuous line) and f_- (dashed line) are shown. At $x = 0.212352$ the f_- solution intersects with the $t = 0$ unphysical solution (dotted line). On the bottom, the free energies for both cases.

depending on the regularization procedure: this actually reflects the particular choice of the fermionic three-dimensional vacuum, that is related to generation of Chern-Simons terms when monopole are present on the sphere. We have presented two opposite choices, both allowed at quantum field theory level, generating different unitary multi-matrix models describing the thermal partition function. The critical behaviors we found, under suitable assumptions on the relevant contributions at small temperature, are very different: in particular we have observed that no Hagedorn transition seems to be present within our second regularization choice. Further studies are surely necessary to elucidate the situation: first of all we expect that supersymmetry should play a role in order to distinguish between the different regularization choices and consistency with the SUSY algebra could probably select a preferred "vacuum charge". On the other hand the relation with the gravitational duals should also be investigated to provide a physical interpretation of the Casimir energies and of the Chern-Simons contributions. Apart from solving the puzzles arisen in this paper, there are a lot of potential interesting developments involving the study of the $\mathcal{N} = 8$ three-dimensional supersymmetric theory considered here. It would be important of course to determine the nature of the phase transition beyond zero 't Hooft coupling and to discuss the issue of exact decoupling limit using chemical potentials, in the spirit of [20, 23, 24]. We also plan to consider the phase diagram in the presence of background scalars as in [55, 56]. More generally one could try to explore if some remnant of four-dimensional integrability persists in three dimensions and to make some quantitative connection, in the strong-coupling limit, between the gauge theory and its gravity dual. It would also be very interesting to study BPS Wilson loops on S^2: in four dimensions there have been exact results for particular classes of loops, the computations reducing to matrix integrals [57, 58] . It is natural to ask if a similar phenomenon takes place in three dimensions too.

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A. Conventions and supersymmetry variations

Before discussing in more details the supersymmetry variations considered in section 2, we shall briefly summarize our conventions and identities on Γ-matrices.

Metric and gauge conventions: The metric is taken diagonal and with Minkowskian signature: $\eta_{MN} = \{-, +, \dots, +\}$. The capital letters $M, N, \dots$ will span the ten dimensional spacetime indices $(0, 1, \dots, 9)$, while the Greek letters $\mu, \nu \dots$ will denote the three dimensional spacetime indices $(0, 1, 2)$. The indices i, j, k are associated to the directions $(1, 2)$ along the sphere S^2, while the directions $(3, \dots, 9)$ transverse to S^2 are indicated with $m, n, \dots$. Finally a special index notation is also reserved to the set of directions $(4, \dots, 9)$ for which we shall use the overlined letters $\bar{m}, \bar{n}, \dots$.

The gauge fields $A = A^a t^a$ are taken to be hermitian and the generator t^a are normalized so that $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. The covariant derivatives are then defined as follows $D_\mu = \nabla_\mu - ig[A_\mu, \cdot]$, where ∇_μ is the *geometrical* covariant derivative. In general we shall omit the trace over the gauge generators in our expressions, unless it is source of confusion.

Some useful Γ-identities: For convenience, here we have collected some Γ-identities, which are useful in checking the supersymmetry invariance of the Lagrangian of our model:

$$\begin{aligned}
&\Gamma^i\Gamma^{jk}\Gamma^i = -2\Gamma^{jk}, && \Gamma^0\Gamma^{jk}\Gamma^0 = -\Gamma^{jk}, && \Gamma^i\Gamma^{0j}\Gamma^i = 0, && \Gamma^0\Gamma^{0j}\Gamma^0 = \Gamma^{0j},\\
&\Gamma^i\Gamma^{jm}\Gamma^i = 0, && \Gamma^0\Gamma^{jm}\Gamma^0 = -\Gamma^{jm}, && \Gamma^i\Gamma^{0m}\Gamma^i = 2\Gamma^{0m}, && \Gamma^0\Gamma^{0m}\Gamma^0 = \Gamma^{0m},\\
&\Gamma^i\Gamma^{mn}\Gamma^i = 2\Gamma^{mn}, && \Gamma^0\Gamma^{mn}\Gamma^0 = -\Gamma^{mn}.
\end{aligned} \tag{A.1}$$

Summation over repeated index is understood. Here Γ^M denotes the ten dimensional matrices, while the symbol γ^μ is used for the three dimensional Dirac matrices. The symbol $\Gamma^{M_1 M_2 \dots M_N}$ defines the completely antisymmetrized product of the matrices Γ^{M_1}, Γ^{M_2},…,Γ^{M_N}.

Three-dimensional fields: The scalar field ϕ^{ij} is antisymmetric in i, j, which are $\mathrm{SU}(4)_R$ indices and it satisfies reality condition:

$$\phi^{ij} \equiv (\phi_{ij})^\dagger = \frac{1}{2}\epsilon^{ijkl}\phi_{kl}. \tag{A.2}$$

It is defined in terms of the old fields $\phi_{\overline{m}}$ by the relations:

$$\begin{aligned} \phi_4 &= \frac{\phi_{14}+\phi_{23}}{\sqrt{2}}, & \phi_5 &= \frac{-\phi_{13}+\phi_{24}}{\sqrt{2}}, & \phi_6 &= \frac{\phi_{12}+\phi_{34}}{\sqrt{2}}, \\ \phi_7 &= i\frac{\phi_{14}-\phi_{23}}{\sqrt{2}}, & \phi_8 &= i\frac{\phi_{13}+\phi_{24}}{\sqrt{2}}, & \phi_9 &= i\frac{-\phi_{12}+\phi_{34}}{\sqrt{2}}. \end{aligned} \tag{A.3}$$

The spinor fields λ_i (again, i is an $\mathrm{SU}(4)_R$ index) denote the Dirac spinors in $D = 3$ originating from the dimensional reduction of ψ_M, while A_μ describes the three-dimensional gauge field.

A.1 Supersymmetry variations

In this appendix, for completeness, we shall write the conditions for the vanishing of the variation at the order α and at the order α^2. At the linear order the complete variation can be summarized by the following table:

Term	**Condition**
$2\mathbb{R}\mathrm{e}\{\alpha g[\phi_{\overline{m}}, \phi_{\overline{n}}]\overline{\psi}\Gamma^{\overline{m}\,\overline{n}}\Gamma^{123}\epsilon\}$	$\mathcal{B} + 2 + P + M = 0$
$2\mathbb{R}\mathrm{e}\{\alpha g[\phi_3, \phi_{\overline{m}}]\overline{\psi}\Gamma^{3\overline{m}}\Gamma^{123}\epsilon\}$	$2\mathcal{B} + 4 + 2P + G - 2M = 0$
$2\mathbb{R}\mathrm{e}\{\alpha i D_0\phi_3\overline{\psi}\Gamma^{03}\Gamma^{123}\epsilon\}$	$4 - 2\mathcal{B} + P + G - 2M = 0$
$2\mathbb{R}\mathrm{e}\{\alpha i D_0\phi_{\overline{m}}\overline{\psi}\Gamma^{0\overline{m}}\Gamma^{123}\epsilon\}$	$4 - 2\mathcal{B} + P + 2M = 0$
$2\mathbb{R}\mathrm{e}\{\alpha i D_i\phi_3\overline{\psi}\Gamma^{i3}\Gamma^{123}\epsilon\}$	$2\mathcal{B} + P + G + 2M + N = 0$
$2\mathbb{R}\mathrm{e}\{\alpha i D_i\phi_{\overline{m}}\overline{\psi}\Gamma^{i\overline{m}}\Gamma^{123}\epsilon\}$	$2\mathcal{B} + P - 2M = 0$
$2\mathbb{R}\mathrm{e}\{\alpha i F_{0i}\overline{\psi}\Gamma^{0i}\Gamma^{123}\epsilon\}$	$-2\mathcal{B} - 2M = 0$
$2\mathbb{R}\mathrm{e}\{\alpha i F_{ij}\overline{\psi}\Gamma^{ij}\Gamma^{123}\epsilon\}$	$\mathcal{B} - 2 + M + \frac{N}{2} = 0$

(A.4)

There are eight different kind of terms, listed in the first column, and they must vanish separately: this leads to the conditions in the second column.

At the quadratic order in α we have simply

Term	**Condition**
$2\mathbb{R}\mathrm{e}\{i\alpha^2\phi_{\overline{m}}\overline{\psi}\Gamma^{\overline{m}}\psi\}$	$-2V + (2 - \frac{\beta}{\alpha})P + MP = 0$
$2\mathbb{R}\mathrm{e}\{i\alpha^2\phi_3\overline{\psi}\Gamma^3\psi\}$	$-2(V+W) + (2 - \frac{\beta}{\alpha})(P+G) - M(P+G) = 0$

(A.5)

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B. Computing the one loop partition function

Here we give all the details of the calculation of the partition function in a monopole background. For the free model the one-loop contribution of each field is a functional determinant, giving the single-particle partition function.

B.1 Computing determinants: the master-formula

We illustrate our regularization scheme: readers who are not interested in these details can take (B.8) and (B.17) as main results, and skip to next subsection.
All the determinants appearing in the evaluation of the *free* partition function contains, as a key ingredient, the evaluation of the following infinite product

$$\Sigma(\eta,\rho,\beta,w) := \prod_{j=0}^{\infty}\prod_{n=-\infty}^{\infty}\left[(j+\eta)^2+\frac{4\pi^2}{\beta^2}\,(n+w)^2\right]^{2j+\rho}. \tag{B.1}$$

This quantity is divergent and it must be regularized. Here, we shall adopt the standard $\zeta-$function regularization and we shall define

$$\Sigma(\eta,\rho,\beta,w) := \mathrm{e}^{-\zeta'(0)}, \tag{B.2}$$

where

$$\zeta(s)=\sum_{j=0}^{\infty}\sum_{n=-\infty}^{\infty}\frac{2j+\rho}{\left[(j+\eta)^2+\frac{4\pi^2}{\beta^2}\,(n+w)^2\right]^s}. \tag{B.3}$$

Notice that (B.3) defines the function $\zeta(s)$ only for $|s|>1$. In order to compute $\zeta'(0)$, we have to consider its analytical continuation to a neighborhood of the origin in the s-plane. This is achieved through a standard technique: firstly, we shall use the Mellin-Barnes representation and subsequently we shall perform a Poisson-resummation in n

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\sum_{j=0}^{\infty}(2j+\rho)\sum_{n=-\infty}^{\infty}\int_0^\infty dt\; t^{s-1}e^{-t(j+\eta)^2-t\frac{4\pi^2}{\beta^2}(n+w)^2}=\\
&=\frac{\beta}{2\sqrt{\pi}\Gamma(s)}\sum_{j=0}^{\infty}(2j+\rho)\sum_{n=-\infty}^{\infty}\int_0^\infty dt\; t^{s-\frac{3}{2}}e^{-t(j+\eta)^2}e^{-\frac{\beta^2n^2}{4t}-2\pi iwn}=\\
&=\frac{\beta\,\Gamma(s-\frac12)}{2\sqrt{\pi}\,\Gamma(s)}\sum_{j=0}^{\infty}\frac{(2\,j+\rho)}{(j+\eta)^{2\,s-1}}+\\
&\quad+\frac{2^{\frac32-s}\beta^{s+\frac12}}{\sqrt{\pi}\Gamma(s)}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\frac{(2j+\rho)}{n^{\frac12-s}(j+\eta)^{s-\frac12}}K_{\frac12-s}(n(j+\eta)\beta)\cos(2n\pi w)=\\
&=\frac{\beta\,\Gamma(s-\frac12)}{2\sqrt{\pi}\,\Gamma(s)}\left(2\zeta(2s-2,\eta)-(2\eta-\rho)\zeta(2s-1,\eta)\right)\\
&\quad+\frac{2^{\frac32-s}\beta^{s+\frac12}}{\sqrt{\pi}\Gamma(s)}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\frac{(2j+\rho)}{n^{\frac12-s}(j+\eta)^{s-\frac12}}K_{\frac12-s}(n(j+\eta)\beta)\cos(2n\pi w).
\end{aligned} \tag{B.4}$$

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The only contribution to $\zeta'(0)$ in (B.4) arises when the derivative acts on $1/\Gamma(s)$ since this quantity vanishes as s approaches 0. We obtain

$$\zeta'(0) = -\beta(2\zeta(-2,\eta) + (\rho - 2\eta)\zeta(-1,\eta)) + \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} \frac{2e^{-n\beta(j+\eta)}(2j+\rho)\cos(2n\pi w)}{n}. \quad \text{(B.5)}$$

From the final expression (B.5) we can deduce two equivalent representations of this result, which are both useful for our goals. Firstly we can perform the sum over j, which yields

$$\zeta'(0) = \beta\left(\frac{2}{3}B_3(\eta) + \frac{1}{2}(\rho - 2\eta)B_2(\eta)\right) + \sum_{n=1}^{\infty} \frac{2x^{n\eta}\left(\rho - x^n(\rho-2)\right)}{n\left(x^n - 1\right)^2}\cos(2n\pi w), \quad \text{(B.6)}$$

with $x := e^{-\beta}$ and $B_k(\eta)$ being the Bernoulli polynomial. Next, we shall define the "single-particle" partition function

$$z_{\text{single}}(x) := \frac{x^{\eta}\left(\rho - x(\rho-2)\right)}{(1-x)^2}, \quad \text{(B.7)}$$

and finally write

$$\log\left(\Sigma(\eta,\rho,\beta,w)\right) = -\beta\left(\frac{2}{3}B_3(\eta) + \frac{1}{2}(\rho - 2\eta)B_2(\eta)\right) - 2\sum_{n=1}^{\infty}\frac{z_{\text{single}}(x^n)}{n}\cos(2n\pi w). \quad \text{(B.8)}$$

This representation will be the most natural when discussing the matrix model and the position of the Hagedorn transition.

Alternatively, in (B.5) we can first sum over n

$$\zeta'(0) = \beta\left(\frac{2}{3}B_3(\eta) + \frac{1}{2}(\rho - 2\eta)B_2(\eta)\right) - \sum_{j=0}^{\infty}(2j+\rho)\left(\log\left(1 - \bar{z}x^j\right) + \log\left(1 - zx^j\right)\right) \quad \text{(B.9)}$$

where $z := e^{-\beta\eta + 2i\pi w}$. If we define

$$\eta(z,x) := \prod_{j=0}^{\infty}\left(1 - zx^j\right) \quad \text{and} \quad \mathcal{M}(z,x) := \prod_{j=0}^{\infty}\left(1 - zx^j\right)^j, \quad \text{(B.10)}$$

we can recast the above result in a very compact form

$$\Sigma(\eta,\rho,\beta,w) = e^{-\beta\left(\frac{2}{3}B_3(\eta) + \frac{1}{2}(\rho-2\eta)B_2(\eta)\right)}\left|\eta(z,q)\right|^{2\rho}\left|\mathcal{M}(z,q)\right|^4. \quad \text{(B.11)}$$

This second representation will be the most suitable when discussing the high temperature behavior. In this limit the leading contribution is encoded in the function $F_\rho(z,x)$

$$F_\rho(z,x) = \sum_{j=0}^{\infty}(2j+\rho)\log\left(1 - zx^j\right). \quad \text{(B.12)}$$

The $x \to 1$ behavior is transparent by rewriting $F_\rho(z,x)$ as

$$F_\rho(z,x) = -\sum_{m=1}^{\infty}\frac{z^m}{m}\left[(\rho-2)\frac{1}{1-x^m} + \frac{2}{(1-x^m)^2}\right], \quad \text{(B.13)}$$

and expanding in β, at fixed z, we get

$$F_\rho(z,x) = -\frac{2}{(\beta)^2}\,\mathrm{Li}_3(z) - \frac{\rho}{\beta}\,\mathrm{Li}_2(z) + \left(\frac{\rho-2}{2}+\frac{5}{6}\right)\log(1-z) + O(\beta). \tag{B.14}$$

To recover (5.13), where the contribution of chemical potentials to the high-temperature limit has been presented, we need further expand $\mathrm{Li}_3(z)$ for $z \to 1$: we are interested in the case when $w = 0$ and $w = 1/2$, appearing respectively in the bosonic and fermionic case, and with zero flat-connection ($z = e^{-y}$, $y \to 0$)

$$\begin{aligned}\mathrm{Li}_3(e^{-y}) &= \zeta(3) - \frac{\pi^2}{6}\,y + \left(\frac{3}{4} - \frac{1}{4}\log y^2\right) y^2 + O(y^3),\\ \mathrm{Li}_3(-e^{-y}) &= -\frac{3}{4}\zeta(3) + \frac{\pi^2}{12}\,y - \frac{1}{4}\log(4)\,y^2 + O(y^3).\end{aligned} \tag{B.15}$$

Fermionic zero modes: In order to compute the contribution of the fermion zero modes, we need to compute the product $\mathfrak{F} = \prod_{n=-\infty}^{\infty}\left[\frac{2\pi}{\beta}(n+w)\right]^\rho$. If we adopt the zeta function regularization as before, we are led to compute the following accessory sum

$$G(s) = \frac{\beta^s}{(2\pi)^s}\sum_{n=-\infty}^{\infty}\frac{\rho}{(n+w)^s} = \frac{\beta^s}{(2\pi)^s}\rho(\zeta(s,w) + e^{i\pi s}\zeta(s,1-w)). \tag{B.16}$$

Then

$$\log(\mathfrak{F}) = -G'(0) = -\rho\sum_{n=1}^{\infty}\frac{e^{-2\pi i n w}}{n}. \tag{B.17}$$

B.2 The scalar determinant

Let us discuss the solution of the eigenvalue problem (4.6). Since our background is static, we can factor out the time-dependence in the eigenfunction by posing $\phi(t,\theta,\phi) \sim \phi_n(\theta,\phi)\; e^{-\frac{2\pi i n}{\beta}t}$. Then the eigenvalue problem in the Weyl basis (4.7) takes the form

$$\begin{aligned}&\sum_{\alpha\in\mathrm{roots}}\left[\frac{4\pi^2}{\beta^2}\left(n+\frac{\beta a_\alpha}{2\pi}\right)^2\phi_{\alpha n} - \hat{\triangle}\phi_{\alpha n} + \frac{\mu^2}{4}\phi_{\alpha n} + \mu^2 q_\alpha^2\phi_{\alpha n}\right]E^\alpha + \\ &+\sum_{i=1}^{N-1}\left(\frac{4\pi^2 n^2}{\beta^2}\phi_{in} + \frac{\mu^2}{4}\phi_{in} - \triangle\phi_{in}\right)H^i = \lambda\sum_{i=1}^{r}\phi_{in}H^i + \lambda\sum_{\alpha\in\mathrm{roots}}\phi_{\alpha n}E^\alpha,\end{aligned} \tag{B.18}$$

where $\triangle$ denotes the geometrical Laplacian for a scalar on the sphere. The symbol $\hat{\triangle}$ instead represents the geometrical Laplacian in the background of a $\mathrm{U}(1)$ magnetic monopole of charge q_α. This Laplacian is constructed with the covariant derivative

$$\hat{D}_\mu = \nabla_\mu - i q_\alpha \mathcal{A}_\mu, \tag{B.19}$$

where ∇_μ is the geometrical covariant derivative. In (B.18) the components along the different directions in the Lie algebra do not interfere and we can consider them as independent. This allows us to split the original eigenvalue problem into two subfamilies, we have: (a) $N(N-1)$ independent eigenvalues coming from each direction along the ladder generator

$$\frac{4\pi^2}{\beta^2}\left(n+\frac{\beta a_\alpha}{2\pi}\right)^2\phi_{\alpha n}-\hat{\triangle}\phi_{\alpha n}+\frac{\mu^2}{4}\phi_{\alpha n}+\mu^2 q_\alpha^2\phi_{\alpha n}=\lambda_{\alpha n}\phi_{\alpha n}, \tag{B.20}$$

and (b) $N-1$ independent eigenvalues coming from the directions along the Cartan subalgebra

$$\frac{4\pi^2 n^2}{\beta^2}\phi_{in}+\frac{\mu^2}{4}\phi_{in}-\triangle\phi_{in}=\lambda_{in}\phi_{in}. \tag{B.21}$$

To begin with, we shall focus our attention on the family (a), since the family (b) can be obtained from (a) as a limiting case for $a_\alpha,\ q_\alpha\to 0$. The solution of the eigenvalue equation (B.20) can be translated into an algebraic problem if we introduce the angular momentum operator in the presence of a U(1) monopole of charge q_α. Its form [59] is

$$L_i^{(\alpha)}=\epsilon_{ijk}x_j(-i\partial_k-q_\alpha A_k)-q_\alpha\frac{x^i}{|x|}\equiv\epsilon_{ijk}x_jP_k-q_\alpha\frac{x^i}{|x|}. \tag{B.22}$$

Here x^i are the Cartesian coordinates of a flat $\mathbb{R}^3$ where our sphere S^2 is embedded. In terms of this auxiliary operator, the kinetic operator in (B.20) takes the form

$$\mu^2(L^{(\alpha)})^2\phi_{\alpha n}+\left[\frac{4\pi^2}{\beta^2}\left(n+\frac{\beta a_\alpha}{2\pi}\right)^2+\frac{\mu^2}{4}\right]\phi_{\alpha n}=\lambda_{\alpha n}\phi_{\alpha n}. \tag{B.23}$$

Thus our task is reduced to finding the eigenvalues and the eigenfunctions of this *dressed* angular momentum operator $(L^{(\alpha)})^2$. Its spectrum[11] was determined thirty years ago by Wu and Yang [59] and it is formally equal to that of the usual angular momentum: the eigenvalues are $j_\alpha(j_\alpha+1)$ and their degeneracy is $2j_\alpha+1$. What changes is the range spanned by the index j_α, which now is $|q_\alpha|,\ |q_\alpha|+1,\ |q_\alpha|+2,\ \cdots$. Putting everything together the spectrum of the kinetic operator (B.20) turns out to be

$$\lambda_{\alpha n}=\mu^2\left(j_\alpha+\frac{1}{2}\right)^2+\frac{4\pi^2}{\beta^2}\left(n+\frac{\beta a_\alpha}{2\pi}\right)^2\quad\text{with}\quad j_\alpha=|q_\alpha|,\ |q_\alpha|+1,\ |q_\alpha|+2\cdots, \tag{B.24}$$

and each eigenvalue has degeneracy $2j_\alpha+1$. Notice that the spectrum does not depend on the sign of q_α. The contribution of the family (a) to the effective action is given by the infinite product

$$\Gamma^{Sc.}{}_{(a)}=\log\left(\prod_{\alpha\in\text{roots}}\prod_{j_\alpha=|q_\alpha|}^{\infty}\prod_{n=-\infty}^{\infty}\left[\mu^2\left(j_\alpha+\frac{1}{2}\right)^2+\frac{4\pi^2}{\beta^2}\left(n+\frac{\beta a_\alpha}{2\pi}\right)^2\right]^{2j_\alpha+1}\right), \tag{B.25}$$

which is easily computed by using the results of appendix B.1 (with $\rho=1+2|q_\alpha|$, $\eta=1/2+|q_\alpha|$, $w=\frac{\beta a_\alpha}{2\pi}$). Setting $x=e^{-\beta\mu}$, we obtain (4.9) and (4.10) The contribution of the family (b) is then obtained from the above results by setting $q_\alpha=a_\alpha=0$.

[11] The eigenfunctions are also known and they are given by the so-called *monopole harmonics* $Y_{qjm}(\theta,\varphi)$. They are a straightforward generalization of the usual spherical harmonics, but we shall not need their explicit form here. We refer the reader to [59] for more details.

B.3 The vector/scalar determinant

The eigenvalue problem for the coupled system (ϕ_3, A) can be simplified by choosing the gauge-fixing (4.11). This choice allows us to cancel some of the mixed terms $(\phi_3 A)$ in the Euclidean quadratic Lagrangian and to obtain

$$\begin{aligned}\mathcal{L}^{(2)}_{(A_\mu,\phi_3)} =& - A_\nu \hat{D}_\mu \hat{D}^\mu A^\nu + R_{\mu\nu} A^\mu A^\nu - i\hat{F}_{\nu\mu}[A^\nu, A^\mu] - [A_\rho, \hat{\phi}_3][A^\rho, \hat{\phi}_3] + \\ &+ \hat{D}_\rho \phi_3 \hat{D}^\rho \phi_3 + \mu^2 \phi_3^2 - [\hat{\phi}_3, \phi_3]^2 - 2\frac{\mu}{\sqrt{g}} \phi_3 \epsilon^{\rho\nu\lambda} k_\rho \hat{D}_\nu A_\lambda.\end{aligned} \tag{B.26}$$

Then, the following coupled eigenvalue problem

$$- \hat{\Box} \phi_3 + \mu^2 \phi_3 + [\hat{\phi}_3, [\hat{\phi}_3, \phi_3]] - \mu\sqrt{g} \epsilon_{\rho\nu\lambda} k^\rho \hat{D}^\nu A^\lambda = \lambda \phi_3, \tag{B.27}$$

$$- \hat{\Box} A_\nu + R_{\mu\nu} A^\mu + i[\hat{F}_{\nu\mu}, A^\mu] + [\hat{\phi}_3, [\hat{\phi}_3, A_\nu]] + \mu\sqrt{g} \epsilon_{\rho\lambda\nu} k^\rho \hat{D}^\lambda \phi_3 = \lambda A_\nu. \tag{B.28}$$

Since both the geometrical and the gauge background are static, the time-component of the vector field $\omega = k^\rho A_\rho = A_0$ decouples completely from the above system. It satisfies the massless version of the scalar equation studied in B.2, namely the eigenvalue problem associated to this component is

$$-\hat{\Box}\, \omega + [\hat{\phi}_3, [\hat{\phi}_3, \omega]] = \lambda_0 \omega. \tag{B.29}$$

We shall forget about ω since its contribution is cancelled by the ghost determinant. We are left with the system given by (B.27) and (B.28) where the indices run only over space.

We expand the coupled system (B.27) and (B.28) in the Weyl basis and we factor out the time-dependence of the eigenfunctions: $A_\mu(t, \theta, \phi) \sim A_{n\mu}(\theta, \phi)\; e^{-\frac{2\pi i n}{\beta} t}$ and $\phi_3(t, \theta, \phi) \sim \phi_{3n}(\theta, \phi)\; e^{-\frac{2\pi i n}{\beta} t}$. Along the directions associated to the ladder operators E_α we find

$$\begin{aligned}-\hat{\triangle} A_{i\alpha n} + m_n^2 A_{i\alpha n} + i\mu^2 q_\alpha \sqrt{g} \epsilon_{ij} A^j_{\alpha n} + \mu^2 q_\alpha^2 A_{i\alpha n} + \mu\sqrt{g} \epsilon_{ji} \hat{D}^j \phi_{3\alpha n} &= \lambda_{\alpha n} A_{i\alpha n}, \\ -\hat{\triangle} \phi_{3\alpha n} + m_n^2 \phi_{3\alpha n} + \mu^2 q_\alpha^2 \phi_{3\alpha n} - \mu\sqrt{g} \epsilon_{ij} \hat{D}^i A^j_{\alpha n} &= \lambda_{\alpha n} \phi_{3\alpha n},\end{aligned} \tag{B.30}$$

where $m_n^2 = (2\pi n/\beta + a_\alpha)^2 + \mu^2$. In the first equation, the symbol $\hat{\triangle}$ denotes the Laplacian on vectors in the background of a monopole of charge q_α, while in the second represents the Laplacian on scalars. Along the Cartan directions we shall again get the system (B.30) but for $q_\alpha = 0$.

To find explicitly the spectrum of system (B.30), it is convenient to decompose our vector $A_{i\alpha n}$ in its selfdual part $A^+_{i\alpha n}$ and anti-selfdual part $A^-_{i\alpha n}$. Consequently we shall introduce the differential operators $O^{(\alpha)}_\pm$ mapping (anti-)selfdual vectors into scalars and their adjoints, mapping scalars into (anti-)selfdual vectors. They are defined by

$$O^{(\alpha)}_\pm V_\pm \equiv O^{i(\alpha)}_\pm V_{\pm i} = \frac{1}{\sqrt{g}} \epsilon^{ij} \hat{D}_i V_{(\pm)j}, \qquad O^{(\alpha)\dagger}_\pm \phi \equiv O^{i(\alpha)\dagger}_\pm \phi = \mp\frac{i}{2} \left(g^{ij} \pm \frac{i}{\sqrt{g}} \epsilon^{ij} \right) \hat{D}_j \phi, \tag{B.31}$$

where $\hat{D}$ as in (B.30) stands for the covariant derivative in the background of a monopole of charge q_α. In terms of these operators, the system (B.30) takes the form

$$\begin{aligned}
O_+^{(\alpha)\dagger}O_+^{(\alpha)}A^+_{\alpha n}+\frac{q_\alpha^2\mu^2}{2}A^+_{\alpha n}+\frac{\ell_n^2}{2}A^+_{\alpha n}-\frac{\mu}{2}O_+^{(\alpha)\dagger}\phi_{3\alpha n} &= \frac{\lambda_{\alpha n}}{2}A^+_{\alpha n},\\
O_-^{(\alpha)\dagger}O_-^{(\alpha)}A^-_{\alpha n}+\frac{q_\alpha^2\mu^2}{2}A^-_{\alpha n}+\frac{\ell_n^2}{2}A^-_{\alpha n}-\frac{\mu}{2}O_-^{(\alpha)\dagger}\phi_{3\alpha n} &= \frac{\lambda_{\alpha n}}{2}A^-_{\alpha n},\\
\frac{1}{2}(O_-^{(\alpha)}O_-^{(\alpha)\dagger}+O_+^{(\alpha)}O_+^{(\alpha)\dagger}+q_\alpha^2\mu^2+\ell_n^2+\mu^2)\phi_{3\alpha n}-\frac{\mu}{2}O_+^{(\alpha)}A^+_{\alpha n}-\frac{\mu}{2}O_-^{(\alpha)}A^-_{\alpha n} &= \frac{\lambda_{\alpha n}}{2}\phi_{3\alpha n}.
\end{aligned}\tag{B.32}$$

Here we have dropped the index i because it is immaterial and we have set $m_n^2=\ell_n^2+\mu^2$. At first sight the eigenvalue problem might appear cumbersome, but in this representation it is quite simple to provide a basis where our problem reduces to diagonalizing an infinite set of three by three matrices. In fact, let us take $q_\alpha\geq 1$[12] and consider the following *basis* for scalars, selfdual and anti-selfdual vectors on the sphere

$$e^{+\alpha}_{jm}=O_+^{(\alpha)\dagger}Y_{q_\alpha jm}\quad\text{for}\quad j\geq q_\alpha+1,\quad e^{-\alpha}_{jm}=O_-^{(\alpha)\dagger}Y_{q_\alpha jm}\quad\text{and}\quad e^{3\alpha}_{jm}=Y_{q_\alpha jm}\quad\text{for}\quad j\geq q_\alpha.\tag{B.33}$$

Here $Y_{q_\alpha jm}$ are the monopole harmonics, namely the eigenfunctions of the angular momentum (B.22). For the anti-selfdual vector we have to add also $2(q_\alpha-1)+1$ elements coming from the zero modes of $O_-^{(\alpha)\dagger}O_-^{(\alpha)}$. We shall denote them as $e^{-\alpha}_{(q_\alpha-1)m}$. For a detailed proof that (B.33) with the addition of the zero modes is a basis, we refer the reader to [60], where the following two useful identities are also shown to hold:

$$O_{(\pm)}^{(\alpha)}O_{(\pm)}^{(\alpha)\dagger}e^{3\alpha}_{jm}=\frac{\mu^2}{2}((L^{(\alpha)})^2-q_\alpha^2\mp q_\alpha)Y^{q_\alpha jm}=\frac{\mu^2}{2}(j(j+1)-q_\alpha^2\mp q_\alpha))e^{3\alpha}_{jm},\tag{B.34}$$

and

$$O_{(\pm)}^{(\alpha)\dagger}O_{(\pm)}^{(\alpha)}e^{\pm\alpha}_{\pm jm}=O_{(\pm)}^{(\alpha)\dagger}O_{(\pm)}^{(\alpha)}O_{(\pm)}^{(\alpha)\dagger}e^{3\alpha}_{\pm jm}=\frac{\mu^2}{2}(j(j+1)-q_\alpha^2\mp q_\alpha))e^{\pm\alpha}_{jm}.\tag{B.35}$$

Because of (B.34) and (B.35) and the definitions (B.33), $e^{\pm\alpha}_{jm}$ and e^3_{jm} for fixed $j\geq q_\alpha+1$ and fixed m generate an invariant three-dimensional linear subspace for the eigenvalue problem (B.32). The original problem can be then separately solved in each subspace, where it reduces to diagonalizing the following three by three matrix

$$\begin{pmatrix}
m_n^2-\mu^2+j(j+1)\mu^2-q_\alpha\mu^2 & 0 & -\mu^2\\
0 & m_n^2-\mu^2+j(j+1)\mu^2+q_\alpha\mu^2 & -\mu^2\\
-\frac{\mu^2}{2}\left(-q_\alpha^2-q_\alpha+j(j+1)\right) & -\frac{\mu^2}{2}\left(-q_\alpha^2+q_\alpha+j(j+1)\right) & m_n^2+j(j+1)\mu^2
\end{pmatrix}.\tag{B.36}$$

The three distinct eigenvalues of this matrix are given by

$$\lambda_+=\ell_n^2+j^2\mu^2,\quad \lambda_-=\ell_n^2+(j+1)^2\mu^2,\quad \lambda_3=\ell_n^2+j(j+1)\mu^2,\qquad\text{with } j\geq q_\alpha+1.\tag{B.37}$$

For $j=q_\alpha$ self-dual vectors do not exist and the invariant subspace is generated only by $e^{-\alpha}_{q_\alpha m}$ and $e^{3\alpha}_{q_\alpha m}$. Instead of (B.36), we have the two by two matrix that is obtained

[12] The case $q_\alpha\leq -1$ is obtained by exchanging the role of self-dual vectors with that of anti-self dual vectors. The value $q_\alpha=\pm1/2$ and $q_\alpha=0$ will be discussed separately.

from (B.36) by dropping the first row and the first column. Its diagonalization produces the following two eigenvalues $\lambda_- = \ell_n^2 + (q_\alpha + 1)^2$ and $\lambda_3 = \ell_n^2 + q_\alpha(q_\alpha + 1)$. Finally, we have to consider $j = q_\alpha - 1$. In this case, we are left with a one-dimensional invariant subspace generated by $e^{-\alpha}_{(q_\alpha-1)m}$. The eigenvalue is simply $\lambda_- = \ell_n^2 + q_\alpha^2$. Summarizing we have $\lambda_- = \ell_n^2 + (j+1)^2\mu^2$ for $j \geq q_\alpha - 1$ and $\lambda_3 = \ell_n^2 + j(j+1)\mu^2$ for $j \geq q_\alpha$ so that we have extended the range of existence of the eigenvalues (B.37). The degeneracy is always $2j + 1$.

In the following we shall neglect the family with eigenvalue λ_3, since its contribution is cancelled by the ghosts. We shall just consider the first two families $\lambda_\pm$, which instead yield the actual vector determinant. The contribution of λ_+ is obtained from the results of appendix B.1 by setting $w = \beta a_\alpha/(2\pi)$, $\eta = q_\alpha + 1$ and $\rho = 2q_\alpha + 3$

$$\Gamma^V_+ = \sum_{\alpha\in\text{roots}} \left(-\frac{1}{12}\beta\mu \left(8q_\alpha^3 + 18q_\alpha^2 + 10q_\alpha + 1\right) - 2\sum_{n=1}^{\infty} \frac{z^{vect.}_{q_\alpha+}(x^n)}{n}\, e^{in\beta a_\alpha} \right), \tag{B.38}$$

with

$$z^{vect.}_{q_\alpha+}(x) = x^{q_\alpha+1}\left[\frac{(3-x)}{(1-x)^2} + \frac{2q_\alpha}{1-x}\right]. \tag{B.39}$$

The contribution of λ_- is instead obtained setting $w = \beta a_\alpha/(2\pi)$, $\eta = q_\alpha$ and $\rho = 2q_\alpha - 1$:

$$\Gamma^V_- = \sum_{\alpha\in\text{roots}} \left(-\frac{1}{12}\beta\mu \left(8q_\alpha^3 - 18q_\alpha^2 + 10q_\alpha - 1\right) - 2\sum_{n=1}^{\infty} \frac{z^{vect.}_{q_\alpha-}(x^n)}{n}\, e^{in\beta a_\alpha} \right), \tag{B.40}$$

with

$$z^{vect.}_{q_\alpha-}(x) = x^{q_\alpha}\left[\frac{x(1+x)}{(1-x)^2} - 1 + \frac{2q_\alpha}{1-x}\right]. \tag{B.41}$$

When adding these two contributions, we obtain (4.12) and (4.13). For what concerns the non-negative values of the monopole charge, there are still two cases to be considered: $q_\alpha = 1/2$ and $q_\alpha = 0$. In both cases, the elements of the basis coming from the additional zero modes of the operator $O_-^{(\alpha)\dagger} O_-^{(\alpha)}$ disappear [60]. For $q_\alpha = 0$, in the basis (B.33) the element $e^{-\alpha}_{jm}$ with $j = q_\alpha = 0$ is absent. The net effect is to reduce the range of the existence of the eigenvalues $\lambda_- = \ell_n^2 + (j+1)^2\mu^2$ to $j \geq q_\alpha$ for $q_\alpha = 0, 1/2$ and of the eigenvalues $\lambda_3 = \ell_n^2 + j(j+1)\mu^2$ to $j \geq 1$ for $q_\alpha = 0$. By recomputing the contribution of λ_-, for $q_\alpha = 1/2$ we get the same results (B.40) and (B.41) when we use the appropriate values for ρ and η in appendix B.1: $\eta = 3/2$ and $\rho = 2$. This does not happen, instead, for $q_\alpha = 0$: by using $\eta = 1$ and $\rho = 1$, we get (4.14).

B.4 The spinor determinant

In determining the contribution to the total partition function of the spinors λ_i, we shall follow closely the steps of the previous appendix. The fermion kinetic operator expanded around the background (3.3) has the following eigenvalue problem

$$-i\gamma^\mu \hat{D}_\mu \lambda + i[\hat{\phi}_3, \lambda] + i\frac{\mu}{4}\gamma^0 \lambda = \rho\lambda, \tag{B.42}$$

where we dropped the $\mathrm{SU}(4)_R$ index since all the components give the same contribution. Expanding the matrix-valued field λ in the Weyl basis and separating the time-dependence we get

$$\lambda = \left(\sum_{\ell=1}^{N-1} \lambda_{\ell n} H^\ell + \sum_{\alpha\in\text{roots}} \lambda_{\alpha n} E^\alpha\right) e^{-\frac{2\pi i}{\beta}(n+\frac{1}{2})t}. \tag{B.43}$$

The only real difference with the scalar and vector cases is that fermions have antiperiodic boundary conditions along the time circle. The usual procedure will, in turn, disentangle the different components along the Lie algebra and it will divide the eigenvalue problem (B.42) into two subfamilies. As in the scalar case, we have: (a) $N(N-1)$ independent eigenvalues coming from each direction along the ladder generator

$$\not{\mathcal{D}}^{(\alpha n)}\lambda_{\alpha n} \equiv -\gamma^0 \left[\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right) + a_\alpha - i\frac{\mu}{4}\right]\lambda_{\alpha n} - i\gamma^i \hat{D}_i\lambda_{\alpha n} + i\mu q_\alpha \lambda_{\alpha n} = \rho_{\alpha n}\lambda_{\alpha n}, \tag{B.44}$$

and (b) $N-1$ independent eigenvalues coming from the directions along the Cartan subalgebra

$$\not{\mathcal{D}}^{(\ell n)}\lambda_{\ell n} \equiv -\gamma^0 \left[\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right) - i\frac{\mu}{4}\right]\lambda_{\ell n} - i\gamma^i \nabla_i \lambda_{\ell n} = \rho_{\ell n}\lambda_{\ell n}. \tag{B.45}$$

In (B.45) the symbol ∇ denotes the geometrical covariant derivative on spinors while $\hat{D}_i$ in (B.44) is the covariant derivative in the background of a $\mathrm{U}(1)$ magnetic monopole of charge q_α, i.e.

$$\hat{D}_i = \partial_i + \frac{i}{2}\Gamma_i^{ab}\Sigma_{ab} - iq_\alpha A_i. \tag{B.46}$$

We shall first consider the family (a). The problem of diagonalizing the operator (B.44) can be solved algebraically by exploiting the unitary transformation $U = e^{\frac{i}{2}\theta\sigma_2} e^{\frac{i}{2}\varphi\sigma_3}$. In fact, after performing this transformation, the operator (B.44) becomes directly related to the total angular momentum $J^{(\alpha)} = L^{(\alpha)} + \frac{\sigma}{2}$ in the monopole background

$$\mathcal{S} \equiv U^\dagger \not{\mathcal{D}}^{(\alpha n)} U = -\left[\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right) + a_\alpha - i\frac{\mu}{4}\right](\sigma\cdot\hat{r}) + \mu\epsilon^{ijk}\hat{r}_i\sigma_j J_k^{(\alpha)} + i\mu q_\alpha. \tag{B.47}$$

Here $\hat{r}$ stands for the usual radial unit vector in three dimensions while σ_i are the Pauli matrices. In (B.47), the operator $\mathcal{S}$ is the sum of three contributions. There is a reduced Dirac operator

$$\mathfrak{D}^{(\alpha)} \equiv \mu\epsilon^{ijk}\hat{r}_i\sigma_j J_k^{(\alpha)} = i\mu\hat{r}\cdot\sigma + \mu\epsilon^{ijk}\hat{r}_i\sigma_j L_k^{(\alpha)}, \tag{B.48}$$

which is the standard two-dimensional massless Dirac operator in the presence of a monopole, but written in an unusual basis. Then we have a "chiral" mass term proportional to $(\sigma\cdot\hat{r})$, which plays the role of the two-dimensional γ_5 (we have in fact $\{(\sigma\cdot\hat{r}), \mathfrak{D}^{(\alpha)}\}=0$). Finally there is a constant shift proportional to the charge q_α.

Now, we can focus our investigation just on the operator (B.48), since the spectrum of (B.47) follows from that of $\mathfrak{D}^{(\alpha)}$. For each eigenfunction ψ of $\mathfrak{D}^{(\alpha)}$ with eigenvalue $\hat{\rho}\neq 0$ there exists another eigenfunction $(\sigma\cdot\hat{r})\psi$ with eigenvalue $-\hat{\rho}$. The possible values of $\hat{\rho}$ can then be computed by considering the eigenvalues of $(\mathfrak{D}^{(\alpha)})^2$. This operator has the following simple form

$$(\mathfrak{D}^{(\alpha)})^2 = \mu^2\left[(J^{(\alpha)})^2 + \frac{1}{4} - q_\alpha^2\right], \tag{B.49}$$

and it is diagonal on the basis $\psi_{jm\pm}$ of the total momentum eigenfunctions, which satisfy $(\sigma\cdot\hat{r})\psi_{jm\pm}=\psi_{jm\mp}$. The eigenvalues are $\hat{\rho}^2_{j\alpha}=\mu^2((j+1/2)^2-q^2_\alpha)$. The positivity of the operator $(\mathfrak{D}^{(\alpha)})^2$ imposes $(j+1/2)^2-q^2_\alpha\geq 0$, and in turn $j\geq|q_\alpha|-\frac{1}{2}$. The degeneracy of each eigenvalue is $2(2j+1)$. On this basis, the operator $\mathfrak{D}^{(\alpha)}$ is also diagonal and it possesses the following spectrum

$$\mathfrak{D}^{(\alpha)}\psi_{jm+}=\hat{\rho}_{j\alpha}\psi_{jm+}\qquad\text{and}\qquad\mathfrak{D}^{(\alpha)}\psi_{jm-}=-\hat{\rho}_{j\alpha}\psi_{jm-}. \tag{B.50}$$

In (B.50) each eigenvalue has degeneracy $(2j+1)$. The above analysis does not directly extend to the kernel of the operator $\mathfrak{D}^{(\alpha)}$, which is obtained for $j=|q_\alpha|-\frac{1}{2}$. These zero-modes can be classified by using the eigenvalues of the operator $(\sigma\cdot\hat{r})$: we shall denote $\nu_\pm$ the number of zero modes with eigenvalue ±1. Then a simple application of the index theorem shows that $\nu_+=|q_\alpha|-q_\alpha$ and $\nu_-=|q_\alpha|+q_\alpha$, namely for positive q_α we have only zero modes with negative chirality and viceversa.

We now turn back to the problem of diagonalizing the operator $\mathcal{S}$ defined in (B.47). The operator $\mathcal{S}$ on the basis provided by the eigenvectors of $\mathfrak{D}^{(\alpha)}$ is not diagonal. However, on the subspace spanned by the eigenfunctions of non-vanishing eigenvalue, it factorizes in an infinite series of two by two matrices. Each matrix acts on the space generated by the eigenfunctions $\psi_{jm\pm}$ and it has the form

$$\begin{pmatrix}\rho_{j\alpha}+i\mu q_\alpha & -\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right)-a_\alpha+i\frac{\mu}{4}\\ -\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right)-a_\alpha+i\frac{\mu}{4} & -\rho_{j\alpha}+i\mu q_\alpha\end{pmatrix}. \tag{B.51}$$

Since we are only interested in the determinant of the operator $\mathcal{S}$, we shall not really need to convert this matrix into a diagonal form, but it is sufficient the evaluate its determinant

$$\mu^2(j_\alpha+1/2)^2+\frac{4\pi^2}{\beta^2}\left(n+\frac{1}{2}+\frac{\beta a_\alpha}{2\pi}-i\frac{\beta\mu}{8\pi}\right)^2\quad\text{with } j_\alpha=|q_\alpha|+\frac{1}{2},|q_\alpha|+\frac{3}{2},\ldots, \tag{B.52}$$

and to recall that there are $2j+1$ determinant with the same value. Then by using the master formula of appendix B.1 (with $\rho=2+2|q_\alpha|$, $\eta=1+|q_\alpha|$ and $w=\frac{1}{2}+\frac{\beta a_\alpha}{2\pi}-i\frac{\beta\mu}{8\pi}$), the contribution of this part of the spectrum gives (4.17) and (4.18). On the kernel of $\mathfrak{D}^{(\alpha)}$, the operator $\mathcal{S}$ is instead diagonal and it has the following spectrum

$$\begin{aligned}\rho_{n\alpha+}&=\frac{2\pi}{\beta}\left[-n-\frac{1}{2}-\frac{\beta a_\alpha}{2\pi}+i\frac{\beta\mu}{8\pi}+i\frac{\beta\mu q_\alpha}{2\pi}\right],\quad\text{with degeneracy } |q_\alpha|+q_\alpha,\\ \rho_{n\alpha-}&=\frac{2\pi}{\beta}\left[n+\frac{1}{2}+\frac{\beta a_\alpha}{2\pi}-i\frac{\beta\mu}{8\pi}+i\frac{\beta\mu q_\alpha}{2\pi}\right],\quad\text{with degeneracy } |q_\alpha|-q_\alpha.\end{aligned} \tag{B.53}$$

Now we have to deal with the regularization ambiguity discussed in section 4.4. In our case, all the different choices for the cuts in the s-plane are encoded in the two following situations:

(i) we regularize the determinants associated to the "*zero-modes*" of negative and positive chirality by choosing opposite cuts in defining the complex power (one on the real positive axis and the other on the real negative axis). With the help of appendix (B.1), we then obtain (4.19);

(ii) we regularize the determinants associated to the "*zero-modes*" of negative and positive chirality by choosing the same cut in defining the complex power. A similar analysis yields (4.20).

The appearance of a Chern-Simons term for case (ii) and the total fermionic contribution to the effective action for both cases are discussed in section 4.4 as well.

C. U(1) truncation of $\mathcal{N} = 4$ super Yang Mills

In this appendix we show that the previous results can be easily recovered from $\mathcal{N} = 4$ super Yang Mills theory on $\mathbb{R} \times S^3$ by a suitable U(1) projection which gives the maximally supersymmetric theory on $\mathbb{R} \times S^2$.

The single-particle partition function in the representation R, $z^R(x)$, is given by

$$z^R(x) = \sum_E x^E, \tag{C.1}$$

where E is the energy eigenvalue subtracted of the Casimir energy, which can be derived for example with the procedure described in the body of the paper. The eigenvalue E can be computed most directly by noting that the Laplacian on the sphere may be written in terms of angular momentum generators which can be diagonalized by means of generalized spherical harmonics on S^3. The isometry group of S^3 is $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ and we will need the spherical harmonics for scalars, vectors and fermions, which will be denoted by $S_{j,m,\bar{m}}(\Omega)$, $V_{j,m,\bar{m}}(\Omega)$ and $F_{j,m,\bar{m}}(\Omega)$, respectively. Here m and $\bar{m}$ are the eigenvalues of J^3 and $\bar{J}^3$ for $\mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_2$ and Ω represents the coordinates of S^3. We follow here the notation of [30]. Having determined the single-particle partition functions on S^3 we may then perform a U(1) projection to derive the single-particle partition functions on S^2. Such projection amounts in a consistent truncation of $\mathcal{N} = 4$ super Yang Mills as discussed in [30], and it can be realized by taking into account that the only modes that actually contribute to the partition function on S^2 are those for which the eigenvalue of $\bar{J}_3$ is equal to half the monopole charge. The projection onto S^2 can thus be performed introducing into the $\mathcal{N} = 4$ partition functions a U(1) projection operator of the form

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{2i\theta(\bar{J}^3 - q)} \tag{C.2}$$

where $2q$ is the integer monopole charge of the BPS vacua on S^2 and as a notation we shall assume $q \geq 0$.

The projection from S^3 to S^2 rescales the radius of the sphere by 1/2 thus giving an S^2 of radius $R = 1/2$.

C.1 Scalars

Scalars on S^3 can be expanded in scalar spherical harmonics $S_{j,m,\bar{m}}(\Omega)$ where m and $\bar{m}$ take the values $-j/2,\ -j/2+1, \ldots,\ j/2-1,\ j/2$. The energy of a scalar on S^3 with radius

$R_{S^3} = 1$, conformally coupled to curvature, is $E = j + 1$. The partition function for a scalar on S^3 then is

$$z_4^{scal.}(x) = \sum_{j=0}^{\infty} \sum_{m=-j/2}^{j/2} \sum_{\bar{m}=-j/2}^{j/2} x^{j+1} = \sum_{j=0}^{\infty} (j+1)^2 x^{j+1} = \frac{x(1+x)}{(1-x)^3} \tag{C.3}$$

where the lower index on z denotes the spacetime dimension.

Inserting the projector (C.2) we easily get the partition function for a scalar on S^2. The scalar partition function in the presence of a monopole of charge q becomes

$$z^{scal.}(x, q) = \sum_{j=0}^{\infty} \sum_{m=-j/2}^{j/2} \sum_{\bar{m}=-j/2}^{j/2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{2i\theta(\bar{m}-q)} x^{j+1}. \tag{C.4}$$

Performing the sums we end up with an integral

$$z^{scal.}(x, q) = \int_0^{\pi} \frac{d\theta}{\pi} \frac{x(1-x^2)\cos(2q\theta)}{(1+x^2-2x\cos\theta)^2}, \tag{C.5}$$

that can be easily done and gives

$$z^{scal.}(x, q) = x^{2q+1} \left[\frac{(1+x^2)}{(1-x^2)^2} + \frac{2q}{1-x^2} \right]. \tag{C.6}$$

We can now reintroduce the appropriate dependence on the radius $R = 1/\mu$. Keeping into account that the partition function (C.6) is defined on an S^2 with radius $R = 1/2$, to get the one with a generic radius $R = 1/\mu$ amounts in simply replacing

$$x^2 \to x \equiv e^{-\beta\mu} \tag{C.7}$$

without having to compute a single determinant.

C.2 Vectors

Vectors on S^3 can be expanded in vector spherical harmonics $V^{\pm}_{j,m,\bar{m}}(\Omega)$ which belong to the representations $(j_1, j_2) = (\frac{j+1}{2}, \frac{j-1}{2})$ and $(j_1, j_2) = (\frac{j-1}{2}, \frac{j+1}{2})$, respectively. The energy for both the representations is given by $E = j + 1$. The partition function on S^3 for the $+$ vector component is then

$$z_{4+}^{vect.}(x) = \sum_{j=1}^{\infty} \sum_{m=-(j+1)/2}^{(j+1)/2} \sum_{\bar{m}=-(j-1)/2}^{(j-1)/2} x^{j+1} = \sum_{j=1}^{\infty} j(j+2) x^{j+1} = \frac{x^2(3-x)}{(1-x)^3}; \tag{C.8}$$

for the $-$ vector component we obviously have the same result $z_+^{vect.}(x) = z_-^{vect.}(x)$ and the sum of these two quantities gives the partition function for a vector on S^3

$$z_4^{vect.}(x) = z_{4+}^{vect.}(x) + z_{4-}^{vect.}(x) = \frac{x^2(6-2x)}{(1-x)^3}.$$

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Inserting now the projector (C.2) into (C.8) and into the analogous one for V^- we get for the $+$ and $-$ vector components respectively

$$
\begin{aligned}
z_+^{vect.}(x,q) &= \sum_{j=1}^{\infty}\sum_{m=-(j+1)/2}^{(j+1)/2}\sum_{\bar{m}=-(j-1)/2}^{(j-1)/2}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{2i\theta(\bar{m}-q)}x^{j+1} \\
&= \int_0^{\pi}\frac{d\theta}{\pi}\frac{\left(3+x^2-4x\cos\theta\right)\cos 2q\theta}{(1+x^2-2x\cos\theta)^2} = x^{2q}\left[\frac{x^2(3-x^2)}{(1-x^2)^2}+2q\frac{x^2}{1-x^2}\right]
\end{aligned}
\tag{C.9}
$$

and

$$
\begin{aligned}
z_-^{vect.}(x,q) &= \sum_{j=1}^{\infty}\sum_{m=-(j-1)/2}^{(j-1)/2}\sum_{\bar{m}=-(j+1)/2}^{(j+1)/2}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{2i\theta(\bar{m}-q)}x^{j+1} \\
&= \int_0^{\pi}\frac{d\theta}{\pi}\frac{\left(1+x^2-4x\cos\theta+2\cos 2\theta\right)\cos 2q\theta}{(1+x^2-2x\cos\theta)^2}.
\end{aligned}
\tag{C.10}
$$

For $q=0$ this integral gives

$$
z_-^{(vec.)}(x,q=0) = \frac{x^2(1+x^2)}{(1-x^2)^2}, \tag{C.11}
$$

and for $q\neq 0$

$$
z_-^{vec.}(x,q) = x^{2q}\left[\frac{x^2(1+x^2)}{(1-x^2)^2}-1+\frac{2q}{1-x^2}\right]. \tag{C.12}
$$

The limit $q\to 0$ is discontinuous, in complete agreement with the computations done in appendix B.3 . Therefore the sums of the $+$ and $-$ vector partition functions for $q\neq 0$ give

$$
z^{vec.}(x,q) = z_+^{vec.}(x,q)+z_-^{vec.}(x,q) = x^{2q}\left[\frac{4x^2}{(1-x^2)^2}-1+2q\left(\frac{1+x^2}{1-x^2}\right)\right] \tag{C.13}
$$

whereas for $q=0$

$$
z_-^{vec.}(x,q=0) = z_+^{vec.}(x,0)+z_-^{vec.}(x,0) = \frac{4x^2}{(1-x^2)^2}. \tag{C.14}
$$

Again, with the substitution (C.7) we immediately get back the results (4.13), (4.14).

C.3 Fermions

Fermions on S^3 can be expanded in spinor spherical harmonics $F^{\pm}_{j,m,\bar{m}}(\Omega)$ which belong to the representations $(j_1,j_2)=(\frac{j}{2},\frac{j-1}{2})$ and $(j_1,j_2)=(\frac{j-1}{2},\frac{j}{2})$, respectively. The energy for both the representations is given by $E=j+1/2$. Therefore on S^3 we get

$$
z_{4+}^{spin.}(x) = \sum_{j=1}^{\infty}\sum_{m=-j/2}^{j/2}\sum_{\bar{m}=-(j-1)/2}^{(j-1)/2}x^{(j+1/2)} = \sum_{j=0}^{\infty}j(j+1)x^{j+1/2} = \frac{2x^{3/2}}{(1-x)^3} \tag{C.15}
$$

for the + fermion component and the same result for the − fermion component. The sum of these two quantities gives the partition function for a fermion on S^3

$$z_4^{\text{spin}}(x) = z_{4+}^{spin.}(x) + z_{4-}^{spin.}(x) = \frac{4x^{3/2}}{(1-x)^3}.$$

Inserting the projector into (C.15) and into the analogous one for F^- one gets the partition functions for a + or − spinor on S^2

$$\begin{aligned} z_+^{\text{spin}}(x,q) &= \sum_{j=1}^{\infty} \sum_{m=-j/2}^{j/2} \sum_{\bar{m}=-(j-1)/2}^{(j-1)/2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{2i\theta(\bar{m}-q)} x^{j+1/2} \\ &= \int_0^{\pi} \frac{d\theta}{\pi} \frac{2x^{3/2}\,(1 - x\cos\theta)\cos 2q\theta}{(1+x^2-2x\cos\theta)^2} = x^{2q}\left[\frac{2x^{3/2}}{(1-x^2)^2} + \frac{2qx^{3/2}}{1-x^2}\right], \end{aligned} \tag{C.16}$$

$$\begin{aligned} z_-^{spin.}(x,q) &= \sum_{j=1}^{\infty} \sum_{m=-(j-1)/2}^{(j-1)/2} \sum_{\bar{m}=-j/2}^{j/2} x^{(j+1)} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{2i\theta(\bar{m}-q)} x^{j+1/2} \\ &= \int_0^{\pi} \frac{d\theta}{\pi} \frac{2x^{3/2}\,(-x + \cos\theta)\cos 2q\theta}{(1+x^2-2x\cos\theta)^2} = x^{2q}\left[\frac{2x^{5/2}}{(1-x^2)^2} + \frac{qx^{1/2}}{1-x^2}\right]. \end{aligned} \tag{C.17}$$

Adding (C.16) and (C.17) we get the partition function for a fermion on S^2 in the non-trivial background

$$z^{spin.}(x,q) = z_+^{\text{spin}}(x,q) + z_-^{spin.}(x,q) = x^{2q}\left[\frac{2x^2(x^{1/2}+x^{-1/2})}{(1-x^2)^2} + 2q\left(\frac{x^{1/2}(1+x)}{1-x^2}\right)\right]. \tag{C.18}$$

With the substitution (C.7) we get back the result (4.18).

The complete partition function for our theory can now be constructed using (4.5). As we showed before the presence of the monopole background (3.4) breaks the original $\mathrm{U}(N)$ invariance to the subgroup $\prod_{I=1}^{k} \mathrm{U}(N_I)$ so that the positive definite charge $2q$, appearing in the single-particle partition functions, is actually a function of the integers labelling the sectors into which the monopole field splits. It can be written here as

$$q \to q_{IJ} = \frac{|n_I - n_J|}{2}.$$

We easily get

$$\mathcal{Z}_A(x) = \int \left[\prod_{I=1}^{k} dU_I\right] \exp\left\{\sum_{I,J=1}^{k} \sum_{n=1}^{\infty} \frac{1}{n}\left[z_B^{IJ}(x^n) + (-1)^{n+1} z_F^{IJ}(x^n)\right] \mathrm{Tr}(U_I^n)\mathrm{Tr}((U_J^\dagger)^n)\right\}. \tag{C.19}$$

Here k is the number of sectors into which the monopole field splits and reintroducing the appropriate dependence on the radius $R = 1/\mu$ with the substitution (C.7), we recover for the bosonic partition function

$$z_B^{IJ}(x,q) = 6x^{q_{IJ}+1/2}\left[\frac{(1+x)}{(1-x)^2} + \frac{2q_{IJ}}{1-x}\right] + x^{q_{IJ}}\left[\frac{4x}{(1-x)^2} - 1 + 2q_{IJ}\left(\frac{1+x}{1-x}\right)\right], \tag{C.20}$$

and for the fermionic one

$$z_F^{IJ}(x,q) = 4x^{q_{IJ}}\left[\frac{2x(x^{1/4}+x^{-1/4})}{(1-x)^2} + 2q_{IJ}\left(\frac{x^{1/4}(1+\sqrt{x})}{1-x}\right)\right]. \tag{C.21}$$

We thus reobtain with a very simple and straightforward procedure the result (4.33), up to the constant (temperature-independent) Casimir contribution. Of course the path integral approach has the advantages of giving to $\mathrm{Tr}(U_I)$ the meaning of matrix holonomy along the thermal circle and of providing an explicit derivation of the Casimir energies.

D. Solving the matrix model

The solution of the matrix model in the presence of a logarithmic interaction has been reduced, in section 7.1, to solve the non-linear differential equation (7.19) with a given set of boundary conditions. Surprisingly, this equation can be explicitly integrated. To achieve this goal, we first express $\rho(t)$ in terms of t and $\rho'(t)$ by means of (7.19)

$$\rho(t) = \frac{64t\rho'(t)^3 - 16\left(p^2+t-1\right)\rho'(t)^2 + p^2}{16\rho'(t)\left(4\rho'(t)-1\right)}. \tag{D.1}$$

Subsequently we take the derivative of with respect to t on both sides. The differential equation (7.19) factorizes into two factors, which can be set separately to zero. In fact, we obtain

$$\left(256t\rho'(t)^4 - 128t\rho'(t)^3 + 16\left(p^2+t-1\right)\rho'(t)^2 - 8p^2\rho'(t) + p^2\right)\rho''(t) = 0, \tag{D.2}$$

which implies

$$\rho''(t) = 0 \quad \Rightarrow \quad \rho(t) = At + B \tag{D.3}$$

and

$$256t\rho'(t)^4 - 128t\rho'(t)^3 + 16\left(p^2+t-1\right)\rho'(t)^2 - 8p^2\rho'(t) + p^2 = 0. \tag{D.4}$$

Consider first (D.3). This solution can only satisfy the boundary condition (s) associated to the strong-coupling region (see section 7.1) and thus it seems the natural candidate to generate $\mathcal{F}_0^s(t,p)$. This implies that the integration constant B is fixed to be $-\frac{1}{2}p(p+1)$. The constant A is instead determined by imposing that (D.3) actually solves (7.19).[13] We obtain

$$\rho_s(t) = \frac{pt}{4(p+1)} - \frac{1}{2}p(p+1). \tag{D.5}$$

The free energy $\mathcal{F}_0^s(t,p)$ is evaluated by integrating (7.16) with the boundary condition (7.8)

$$\mathcal{F}_0^s(t,p) = -\frac{1}{2}\left((\log(4)-3)p + (p+1)^2\log(p+1) - p^2\log(p)\right) + \frac{t}{4(1+p)} - \frac{p}{2}\log(t)\,. \tag{D.6}$$

As discussed in section 7.1, (D.6) is not the right solution at small t because cannot reproduce the series obtained from the large p expansions. We come now to (D.4). It is an

[13] Since we have taken a derivative of 7.19, we could have potentially added spurious solutions

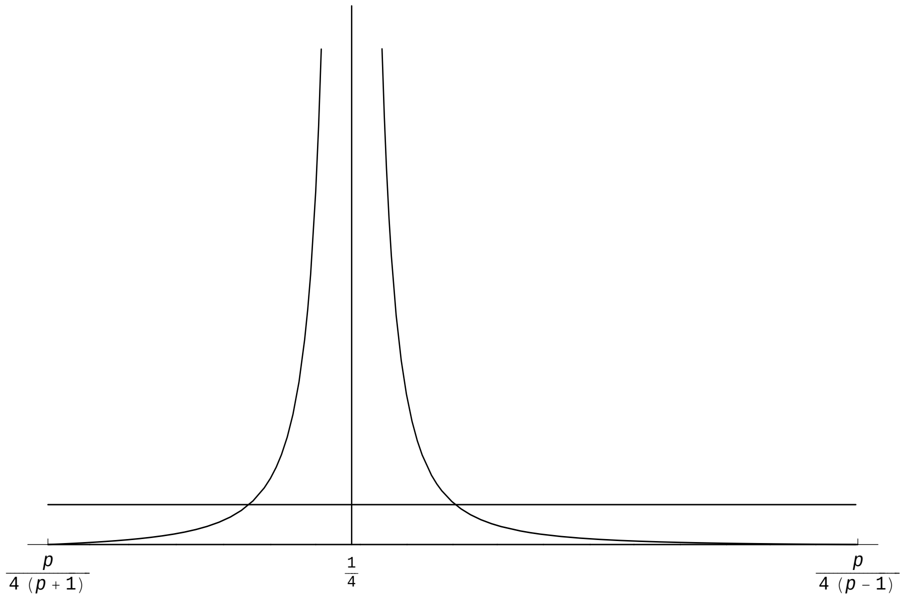

Figure 6: Plot of the r.h.s of (D.7). It diverges for $\rho' = 1/4$. For any positive t we have two solutions.

algebraic quartic equation, which determines $\rho'(t)$ as a function of t and p. We have four solutions, whose qualitative behavior can be investigated by writing the inverse function

$$t = -\frac{(4(p-1)\rho'(t) - p)\,(4(1+p)\rho'(t) - p)}{16\rho'(t)^2\,(4\rho'(t) - 1)^2} \tag{D.7}$$

and by drawing its plot. Since we are interested in positive t and in real solutions, we can focus our attention just on the interval $[\frac{p}{4(p+1)}, \frac{p}{4(p-1)}]$. The plot is given in figure 6. We immediately recognize that there are two potential solutions in this region. At small t, they are both finite and their values at $t = 0$ are respectively

$$\rho_1' = \frac{p}{4(p+1)} \quad \text{and} \quad \rho_2' = \frac{p}{4(p-1)}. \tag{D.8}$$

For large t, both solutions approach $1/4$ but with opposite subleading term. In fact, by setting $\rho \sim 1/4 + bt^\alpha$ in (D.7), we immediately find

$$\rho_1'(t) = \frac{1}{4} - \frac{1}{4\sqrt{t}} + O(t) \quad \text{and} \quad \rho_2'(t) = \frac{1}{4} + \frac{1}{4\sqrt{t}} + O(t). \tag{D.9}$$

The actual functions $\rho_{1,2}(t)$ can be easily recovered by exploiting (D.1), which provides ρ in terms of ρ' and t (and p). It is easy to check that the solution $\rho_2(t)$ can be dropped since its behavior at small and large t is in contrast with the boundary conditions. Instead, we can identify $\rho_1(t)$ with the weak-coupling solution $\rho_w(t)$ and by integrating (7.20) to evaluate $\mathcal{F}_0^w(t,p)$. The integration constant is fixed by requiring that our free energy coincides with that of the Gross-Witten model for large t. The logarithmic interaction is in fact sub-leading for $t \gg 1$. Nicely the integration over t can be performed without an explicit knowledge of $\rho_w(t)$. In fact (D.7) defines an invertible mapping in the range $\frac{p}{4(p+1)} \leq \rho' \leq \frac{1}{4}$ (see figure 6). Thus, by means of (D.7), we can write

$$\begin{aligned}
\mathcal{F}_0^w(t,p) &= f_w + \int dt \left(\frac{1}{4} - \frac{p^2}{2t} - \frac{\rho_w(t)}{t}\right) = \\
&= f_w + \int d\rho'_w \frac{\left(p^2 (4\rho'_w - 1)^3 - 64 (\rho'_w)^3\right)\left(p^2 (4\rho'_w - 1)^3 + 16 (\rho'_w)^2 (4\rho'_w + 1)\right)}{32 (1 - 4\rho'_w)^2 (\rho'_w)^3 \left(p^2 (1 - 4\rho'_w)^2 - 16 (\rho'_w)^2\right)} = \\
&= f_w + \frac{1}{32}\left(\frac{8p^2}{\rho'_w} - \frac{p^2}{2(\rho'_w)^2} + 16\left(\log(\rho'_w)\, p^2 - 2p\tanh^{-1}\left(p + 4\left(\frac{1}{p} - p\right)\rho'_w\right) + \right.\right. \\
&\quad \left.\left. + \log(1 - 4\rho'_w) + \frac{2}{1 - 4\rho'_w}\right)\right).
\end{aligned} \tag{D.10}$$

Here f_w is the arbitrary constant of integration. Requiring that we reobtain the usual Gross-Witten model for $t \gg 1$ fixes our constant to be

$$f_w = -\frac{3}{4} + \frac{1}{4}p((-3 + \log(16))p - 2\log(p-1) + 2\log(p+1)). \tag{D.11}$$

With this choice expansion of the free energy $\mathcal{F}_0^w(t,p)$ for large t takes the form

$$\begin{aligned}
\mathcal{F}_0^w(t,p) =& \sqrt{t} + \frac{1}{4}\left(\log\left(\frac{1}{t}\right) - 3\right) - \frac{1}{2}p^2\sqrt{\frac{1}{t}} - \frac{p^2}{4t} + \frac{1}{24}p^2 (p^2 - 4)\left(\frac{1}{t}\right)^{3/2} + \\
&+ \frac{1}{8}p^2 (p^2 - 1)\left(\frac{1}{t}\right)^2 - \frac{1}{80}\left(p^2 (p^4 - 20p^2 + 8)\right)\left(\frac{1}{t}\right)^{5/2} + \mathcal{O}\left(\frac{1}{t^3}\right).
\end{aligned} \tag{D.12}$$

The leading behavior is independent of p and it coincides with that of the Gross-Witten model. The above expression contains also the result of the semiclassical approximation (7.15), up to higher orders in $p^{2n}/t^{n+m/2}$. We can also compute the small t behavior of this solution and it is given by

$$\begin{aligned}
\mathcal{F}_0^w(t,p) =& \frac{1}{2}\left(\log(p)p^2 + (3 - \log 4)\, p - (p+1)^2 \log(p+1)\right) + \\
&+ \frac{p}{2}\log(t) + \frac{t}{4p+4} - \frac{pt^2}{32(p+1)^4} + \frac{(p-1)pt^3}{96(p+1)^7} + O\left(t^4\right).
\end{aligned} \tag{D.13}$$

Surprisingly, we see that $\mathcal{F}_0^w(t,p)$ satisfies also the boundary condition (7.8) for small t and reproduces, in that regime, the result of the large p expansion. In other words, (D.10) and (D.11) provide a solution which smoothly interpolates between the strong and the weak coupling regime.

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